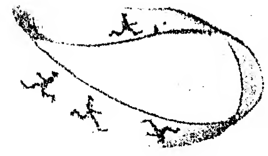
PLAY THEMES



TA'L'KING CHANCES

Introducing a new series on mind games

"WHEN a pair of standard dice is thrown, 11 and 12 have equal chances to turn up because these numbers can arise in only one combination ($12=6+6$; $11=6+5$; unlike, for example, $10=6+4=5+5$)". If you agree, you are in good company; the great French mathematician, Leibnitz, thought so. Of course, the argument is wrong!

Probability theory is indeed a goldmine for recreational mathematicians. It abounds with paradoxical situations of every kind, most of which are much more subtle than the one given above. Before looking at some of these examples, let us briefly review the basic concepts of probability. Consider a well-defined situation in which a particular event can happen in h ways and fail to happen in f ways, there being no other possibility. Then the probability for the event to occur is $h/(h+f)$, *provided all the different (h+f) outcomes are equally likely*. (We assume that we have some pragmatic way of knowing whether the outcomes are equally likely.) This basic rule is supplemented by the principles of addition and multiplication of probabilities: (a) if two independent events (that is, events that do not influence each other) have probabilities of occurrence P_1 and P_2, then the probability for *both* to happen is $P_1 \times P_2$; and (b) if two mutually exclusive events have probabilities P_1 and P_2 then the probability that at least one of them will occur is (P_1+P_2). These rules are quite straightforward; the trouble arises in making sure that the basic conditions under which the rules work are satisfied in a problem.

Let us apply these rules to some simple cases. A coin is tossed 9 times and it turned up heads every time. If we toss it again, what is the probability that it will turn up 'heads'? Just ½; the incredible run of 9 heads in no way influences the coin to favour 'tails' at the 10th toss. On the other hand, if a coin is tossed ten times, what is the probability of its turning up "heads" ten times? The answer is, ½×½... multiplied ten times, or (1/1024). The distinction between the two cases is elementary but noteworthy.

A closely related theme is involved in a rather well-known paradox. Three coins are tossed simultaneously. What is the probability that all of them will turn up alike? We know for sure that two out of three coins will turn up alike. The third can either be the same as the two or different. Since these are the only two possibilities, the probability is ½. But we can also argue as follows:

> How good are you at chance games? ... Try and take your chance

The probability for all the three to be 'heads' is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=1/8$ and the probability for all three to be 'tails' is 1/8. Thus the probability to get *either* all head *or* all tail is $(1/8+1/8)$ which is ¼. Which is the correct solution?

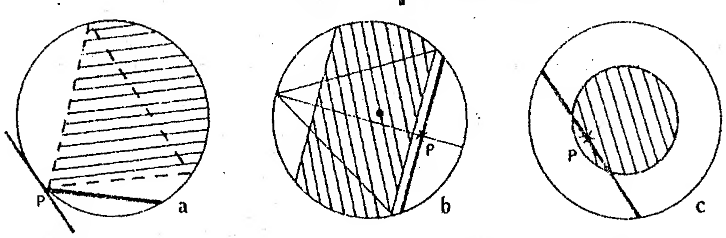
Such fallacious reasonings can seemingly help one to achieve the impossible. Consider the following impossibility: A bag contains two counters of which nothing is known except that each is either black or white. Ascertain their colours by pure reasoning! Lewis Carroll, the author of *Alice in Wonderland* and a puzzlist, gave the following argument to "prove" that the bag must have one black and one white counter:

"We know that the possibilities are BB,BW,WB,WW (B-black, W-white) each with probability ¼. Consider the following 'thought experiment': We add a black counter to the box (changing the possible configurations to BBB, BWB, WBB, WWB), shake it well, and pick out one counter at random. The chance for this counter to be black is easily computed to be $1\times\frac{1}{4}+2/3\times\frac{1}{4}+2/3\times\frac{1}{4}+1/3\times\frac{1}{4}=2/3$. We have here a situation in which the probability of drawing a black counter from a box containing three counters, to be 2/3. This can happen only if the box has 2 black and 1 white counters. So, before we added the black counter, the box must have had one black and one white couner." An attentive reader can easily spot the flaw (see p. 36).

The paradoxes mentioned above arise from faulty reasoning. Probability theory is also full of examples where the results *appear* to be paradoxical (that is, counter to our normal intuition) though the reasoning is quite correct. Two famous examples are the Bertrand's paradox and the St. Petersburg paradox (see boxes); the first arises from the ambiguous statement of the problem while the second requires extra constraint to be introduced for an intuitively acceptable solution.

Another class of counter-intuitive results crops up when one tries to choose the best course of action based on probability estimates. Consider the following situation: There are 3 machines A,B and C, each having a TV screen and a button. On pressing the button, A will always display number 3 on the screen; B will display 2 or 4 or 6 with probabilities 0.56, 0.22 and 0.22, respectively; C will display 1 or 5 with probabilities 0.51 and 0.49. You are allowed to choose one of the three machines and your friend will choose another. Both press the buttons and the one who gets the larger number on display wins. What is your best

Bertrand's paradox



IF a chord is drawn at random in a circle, what is the probability that the chord is longer than the length of the inscribed equilateral triangle in the circle?

It turns out that the answer depends on how exactly the chord is chosen. Figure shows three different ways of choosing the chord each of which gives a different probability. In (a), we choose some point P and draw the chord in a random direction. Since only one-third of possible directions produce the desired result, the probability is 1/3. In (b), we choose an arbitrary diameter and an arbitrary point in it, and draw the chord perpendicular to the diameter at that point. Clearly, half the points in any diameter will produce chords of the required length, giving a probability of ½. Yet another possibility is to choose a point randomly inside the circle and draw a chord with that chosen point as middle point. As C shows, the probability is now ¼!

T. P.

choice, A,B or C? This question is easy to answer. We see that A (displaying 3) beats B whenever B comes up with 2, which happens 56 per cent of the time. Thus A beats B with a probability 0.56. Also A beats C whenever C comes up with 1 which happens 51 per cent of the time. Thus

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choosing A will give you an edge over your friend. We can also calculate the probability with which B beats C; it is (1×0.22) + (0.22×0.51) + (0.56×0.51) = 0.6178. Clearly A beats B, B beats C and A beats C. A is the best; C the worst.

Now suppose the same game is played with three players. Each presses the button and the one who comes up with the highest number wins. If you are to make the first choice, will you still choose A? Intuition would say that addition of a player should not affect the 'fact' that A is the best choice; and intuition would be wrong. Now C is the best choice! As one can compute, A wins with a probability of $0.56\times0.51=0.2856$ (B showing 2; C showing 1); B wins with the probability of $(0.44\times0.51)+(0.22\times0.49)=0.3322$ (B with 4 or 6 and C with 1, or B with 6 and C with 5) and C wins with the probability of $0.49\times0.78 = 0.3822$ (C with 5 and B with 2 or 4). Thus C has an edge over A and B.

There is nothing wrong in the analysis above. The "paradox" was invented by Colin R. Blyth and has many variants. It is a fact of life that crops up in probability-based decision-making.

Those who think such a game is too contrived will be surprised by the following practical situation pointed out by Martin Gardner. Suppose the effectiveness of three drugs A,B and C against a disease is graded in a scale of 1 to 6. Further, suppose that tests on patients show the effectiveness to be distributed with the same probability as discussed above (that is, A is uniformly effective at level 3; C produces level 1, 51 per cent of time, etc). If *only A and C* were in the market, any doctor will prefer A to C. But the moment B is introduced, the doctor probably should prefer C to A!

There are other situations in which the concepts of probability go against one's intuitive expectations. Suppose we are faced with three choices A,B and C. If (probabilistically) A is better than B, and B is better than C, then we expect A to be better than C, thereby defining a 'best' choice. This may not be always the case. One notorious sucker bet which professional gamblers pull on unsuspecting victims is based on this fact. From a deck of cards, the following nine cards are chosen: ace, six and eight of spades; three, five and seven of hearts; and two, four and nine of clubs. The three spade cards are shuffled and kept face down in the top row (row A); hearts in the middle row (row B) and clubs in the bottom row (row C). The sucker is allowed to take a card from any row and then you take a card from a different row. The higher card wins. The reader can convince himself that from whichever row the sucker picks up his card, you can always find another row which will give you winning odds of four to five. In the arrangement given above A beats B, B beats C *and* C beats A! There just isn't a best row to choose.

We conclude with three questions (send your answers to SCIENCE TODAY).

(1) James and Ramesh, meeting at a pub, are bragging as to who has more money in his purse. They decide to settle the matter by explicit counting. It is also agreed that the one who wins (that is, the one who has more money) will give all the money he has to the loser as a mark of consolation. James thinks, "Either I win or I lose. If I win, I lose my money; but if I lose, I get his purse which has *more* money. So the bet favours me." Well, Ramesh is thinking the same thing! How can a bet favour both the parties?

(2) Two black aces and one red ace are taken from a deck of cards, shuffled and dealt to A, B and C. A fourth person D, peeps at the hands of A, B and C and turns over a black ace from the hands of B or C (that leaves one black and one red ace). At this stage, what is the probability that A has the red ace?

(3) A,B and C agree to fight a duel on following rules: (i) they draw lots to decide who fires first, second and third; (ii) they will fire single shots by turn and continue in the same cyclic order until two are dead; and (iii) at each trial, the man can aim anywhere he pleases. All three know that A never misses, B hits 80 per cent of the time and C hits 50 per cent. Assuming all adopt the best strategy, what are the chances of survival of A,B and C?

T. Padmanabhan

Dr. Padmanabhan is with the Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Bombay. He has a keen interest in recreational mathematics.

St. Petersburg paradox

IN order to understand this paradox, we need to know the concept of "expectation" in a bet (or contest, or game). Suppose A and B decide to play the following game: A rolls a dice and pays B as many rupees as the figure that turns up (that is, Re 1 if the dice shows 1; Rs. 2 if it is 2, etc). On the *average*, how much money does B expect to get per game? Since the probability for him to get Re. 1 is 1/6, for Rs. 2 is 1/6, etc, on the *average*, he expects to get $1/6\times1+1/6\times2 \ldots 1/6\times6=3.5$ rupees (if he plays 6000 games, he can expect to get Re 1, thousand times; Rs. 2, thousand times, and so on, giving a total of Rs. 21,000 in 6000 games. It averages to Rs. 3.5 per game). Thus if we want to be fair, we can insist on B paying A, Rs. 3.5 for the privilege of playing each game.

The St. Petersburg paradox arises in the following game devised by J. Bernoulli. A tosses a coin and pays B Re.1 if it turns up 'heads' in the first trial. If it is 'tails', A tosses it again, and pays B *Rs. 2* if it is 'heads' on the second trial. Otherwise, he tosses again, and pays Rs.4 if it is 'heads' on the third trial and so on. In short, A continues to toss until the head appears on the n^{th} trial; then he pays B, Rs.2^{n-1} (that is, $2\times2\times\ldots$ multiplied n times). Now, how much can B expect to get, on the average?

The probability that B will get Re.1 is ½, Rs. 2 is ½×½, Rs. 4 is ½×½×½, etc. Thus his expectation will be Rs. $(1\times\frac{1}{2}+2\times\frac{1}{2}\times\frac{1}{2}+4\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\ldots)$ This is same as the sum Rs. (½+½+½..) which is clearly infinite! Thus B can expect to gain infinite wealth from such a game, even though in any single game, A will pay B only a finite amount! (Also note that, to make the game fair, B has to pay A an infinite amount).

Just as in Bertrand's paradox, there is no clear-cut resolution to the above dilemma. The usually accepted answer is the following: In a practical situation, A has only a finite amount of wealth, however large. Thus A just cannot pay B any amount larger than that limit. Let us suppose, to make the calculation easy, that A has Rs. 2^{20}, that is, Rs. 10,48,576. Then the game can at most go upto 21 tosses of coin. (If no heads appear until then, A gives B all his money, shakes hands and quits.) Now B's expectation is quite finite; it is (½+½... 21 times) which is Rs. 10.50! So if B pays A Rs. 10.50 the game is fair. Quite a reasonable amount.

T. P.

GAMES(WO)MANSHIP

Of logical games and strategic play

TWO-PERSON games are an integral part of recreational mathematics. They are popular because they seldom require formal training in mathematics.

Normally, two-person games of logic satisfy the following conditions: (i) the game ends after a finite number of moves, (ii) no random ('chance') device, such as dice or cards is introduced to decide the moves, and (iii) both the players have full information about the moves played up to any point. The 'rules' and 'moves' in the game need not satisfy any other conditions. In games of this type, if both the players play "rationally", that is, by choosing the best move at each stage, the outcome *is predetermined*. The question, of course, is to analyse the game and find out the best course of play ("strategy") for each player. As we shall see, it may not always be simple.

Consider, for example, the following game. Two players take turns in placing circular discs of radius 1 cm anywhere on a table of radius, say, 50 cm. Each disc must be put down flat, within the border of the table and without moving any previously placed disc. The player who cannot find space to put a disc loses the game.

It is easy to see that the player who has the first move can always win by a technique called 'pairing strategy'. He places the first disc in the centre, and thereafter duplicates his opponent's move by choosing the symmetrically opposite point (with respect to the centre of the table) to place his disc. Clearly, as long as his opponent can find a place, so can he.

Very few games afford such a simple analysis. What is more, games which are superficially similar often end up having very different strategies. Described here are some games which superficially resemble the games of tick-tack-toe (noughts and crosses), Nim and Hex (see box) followed by some questions for the reader to answer.

Consider first the following 'three-together' game played in one dimension. The "board" is a single row of an even number of squares or cells. (Fig.1 shows the case with 10 squares). Two players A and B take turns placing counters, one at a time, in any of the cells. The counters are all alike; there is no distinction between A's

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counters and B's. The player who gets three counters in adjacent cells is the winner. (Fig. 1 shows a position which is a loss for the player who has to move; wherever he puts the counter, the other player gets three adjacent cells filled in his turn.)

The game looks surprisingly simple and has a superficial similarity with the "three-in-a-row" of tick-tack-toe while, actually, there is no similarity at all. For most cases, the first player seems to win, but not always. As far as I know, no simple strategy or rule has ever been discovered. The reader can amuse himself by analysing the game with four squares which is very simple and six squares which is not so simple. The game was extended to two dimensions by the noted puzzlist, Stainislaw Ulam. The board is now a square, with an even number of cells on the sides and the first player to get three in a row orthogonally or diagonally is the winner. Here again, it is simple to analyse the game when the board is a 4×4 square but not so when it is 6×6 or more. Also, can the reader figure out why the above two versions are played on boards with an even number of sides? (See p.74)

Let us now look at the variants of the game Nim. One simple variant is called the Prime Nim (First analysed by C.E. Shannon) in which players are allowed to diminish the heaps only by prime numbers (counting 1 also as a prime number). Surprisingly, the standard Nim strategy can still be used with the following modification: one should regard each heap as being equal to the remainder when the number is divided by 4 (since 4 is the first non-prime number) and adopt the usual Nim strategy.

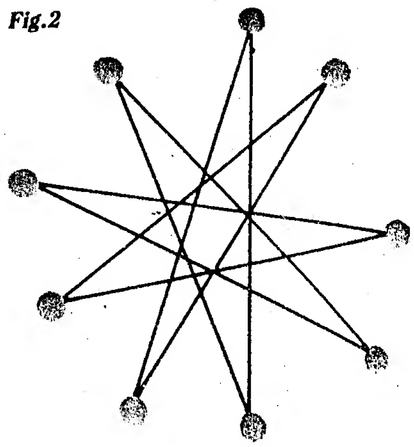
Fig.1

The reader can easily convince himself in a few trials as to how this method works.

Another modification of Nim consists of placing counters at the vertices of a planar figure. Suppose counters are placed on the nine tips of the star shown in Fig. 2. And A and B take turns removing either one counter or two counters connected by the straight lines. The player to take the last counter wins. Can the reader find out who will win and how? The analysis is relatively simple.

There are also numerous games which appear similar to Hex, the most notorious variation being the one in which the sides of the Hex board differ by one unit, that is, instead of an n×n, one uses a n x (n+1) board. The second player who plays between

Fig.2



the closer edges can always win using a simple pairing strategy.

Another version called Bridge-It (or Gale, after the inventor, David Gale) is played on a board with alternating mauve and black dots as shown in Fig. 3. One player uses a black pencil and connects any pair of adjacent black spots. The second player uses a colour pencil and connects mauve dots. No line should cross another. The one who manages a connected path joining two opposite sides of his colour is the winner. It is easy to show that the first player can win but the strategy itself is not so easy to spot.

And while the reader has the two pencils, black and colour, there are two more games that he can play. In the first, two players start with six dots in the form of a regular hexagon (Fig. 4). They alternately connect a pair of dots, one using the colour pencil and other, black. The player who is forced to form a triangle of his own colour is the loser (of course, only those triangles which have vertices on the original six points

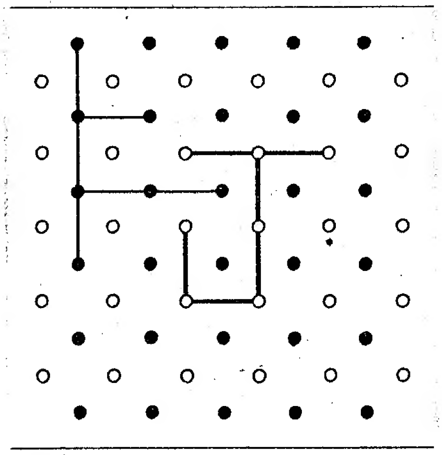


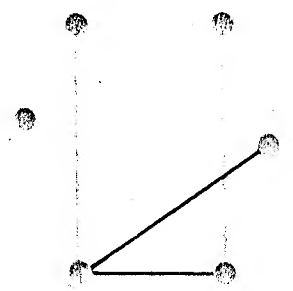
Fig.3

count). It is straightforward to show that a drawm game is impossible (can you do it?). Exhaustive computer analysis by G.J. Simmons, the inventor of the game, showed that the second player can always win.

The second game is played on any n×n lattice of points (Fig.5). Pairs of opposite sides are identified as 'mauve' and 'black'. Players take turns connecting two adjacent dots. Mauve's aim is to construct a path connecting the 'mauve' sides; Black's is to construct a path connecting the other pair of sides (opposing paths may cross each other at right angles).

The superficial resemblence to Hex is misleading because this game, unlike Hex, can be drawn. Played on a 3×3 lattice, (that is,3 points on each side), the game is an easy win for the first player. To analyse 4×4 is difficult but it can also be shown to be a win for the first player. Can the reader spot

Fig. 4



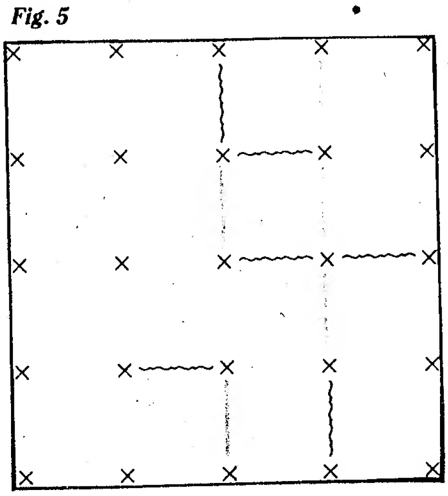
the winning first move? On higher boards, it is not known as to who has the advantage, or whether it is a draw.

We shall conclude with three games which the reader can analyse. Answers as well as any interesting points communicated (to SCIENCE TODAY) by readers will be presented in a future issue.

(1) From a standard deck of cards, select 1 (ace), 2, ...9 (nine cards) of a single suit (say, spades) and keep them face up on the table. Two players alternately pick up cards from the table, one at a time. The first person to pick up cards that total 15 is the winner. If both players play rationally, what is the outcome?

(2) Eight black pawns and eight white pawns are kept on the standard (8×8) chessboard according to the following prescriptions: (i) one white and one black pawn should be in each vertical column, (ii) the black pawn must be to the "north" of the white pawn in each column. (see Fig. 6 for a typical position). White pawns move up and black pawns move down as many squares as desired. The pawns cannot 'jump over'

Fig. 5



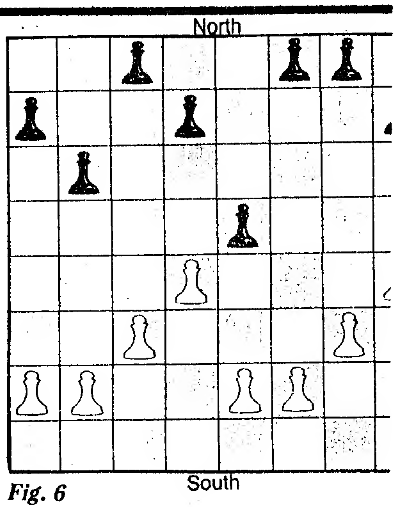


Fig. 6

pawns; also there is no capture. Two play (black and white) alternately move th pawns. The player who gets all his pav blocked (and hence cannot move) is loser. Can the reader analyse the garr

(3)Discusing the game of Bridge earlier, I had said it is easy to show that first player can win but the strategy itsel not so easy to spot. Can the reader find strategy?

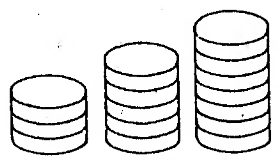
T. Padmanabl

Dr. Padmanabhan is with the Theore Astrophysics Group, Tata Institute of I damental Research, Bombay. He has a i interest in recreational mathematics.

Games of Nim and Hex

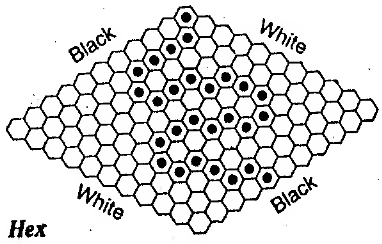
IN the game of Nim, one starts with a set of rows of counters, each containing any specified number of counters. For example, one can start with 3 rows, containing, say, 3, 5 and 7 counters, as shown in the figure. Two players take turns removing as many counters as they want from any *single* row, that is, they cannot take a counter from the first row and another from the second. Whoever takes the last counter wins. (The name Nim probably comes from an old English word, meaning "to take away" or "steal".)

The strategy to win at Nim is simple. Write down the number of counters in each row as the sum of powers of two (eg 3=2+1; 5=4+1; 7=4+2+1) and cross out the powers that appear in pairs (eg $3=\cancel{2}+\cancel{1}$; $5=\cancel{4}+\cancel{1}$; $7=\cancel{4}+\cancel{2}+1$; two 4's two 2's and two 1's are crossed out, leaving 1). The 'safe' position for a player is the one in which there is no left-over (eg here the position is 'unsafe' because of the left-over, 1). So one should always remove counters so as to convert an unsafe position to a safe one. (eg, here removing one counter from any heap will do it).



Nim

The game of Hex is played on an n x n board of hexagons as shown in the figure here (here n=11). Two opposite sides are labelled "black" and the other two "white". (The corners belong to either side.) One player (B) has a supply of



Hex

black counters while the other player (W) uses white counters. They take turns placing counters in any unoccupied cell. B tries to make a chain of black counters between the 'black' sides while W tries for a similar white chain between the white sides. No winning strategy is known, though it can be proved that a winning strategy *exists* for the first player.

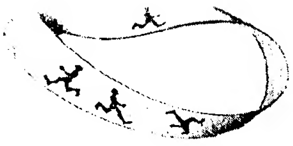


FIGURE IT OUT

SUPPOSE a square piece of paper is cut into smaller pieces and rearranged to form another figure. Can the area of the figure change? "No," one would say; but that is precisely what seems to happen when a square of 8 cm side is cut and rearranged. In Fig. 1a, the square has an area 64 square cm. When this figure is cut along the red lines and the

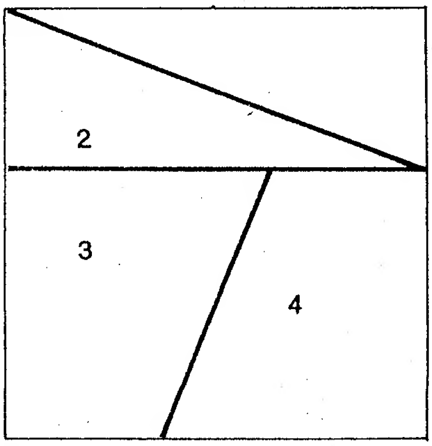


Fig. 1a

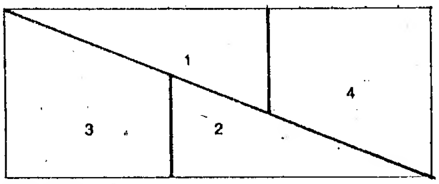
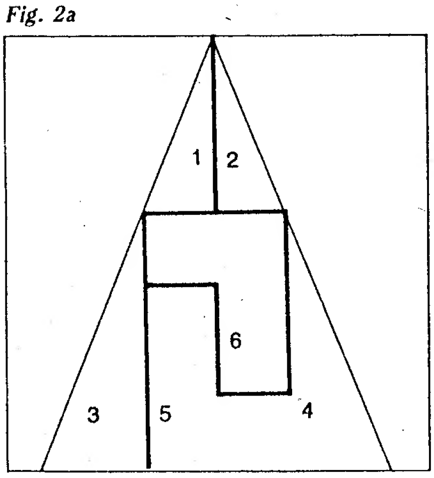


Fig. 1b

parts are rearranged to form a rectangle of length 13 cm and width 5 cm as in Fig. 1b, the area has increased to 65 square cm! Fig. 2 shows a similar rearrangement of a triangle of 60 sq. cm into one of 58 sq. cm, because of the hole that develops in the centre. (The reader can easily construct

Fig. 2a



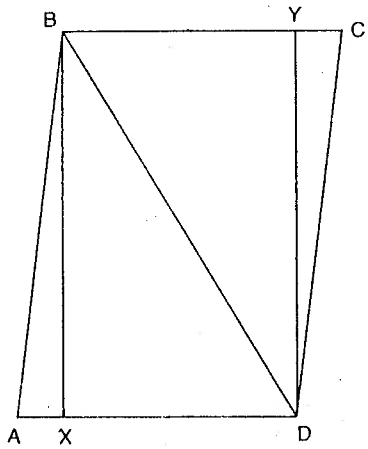


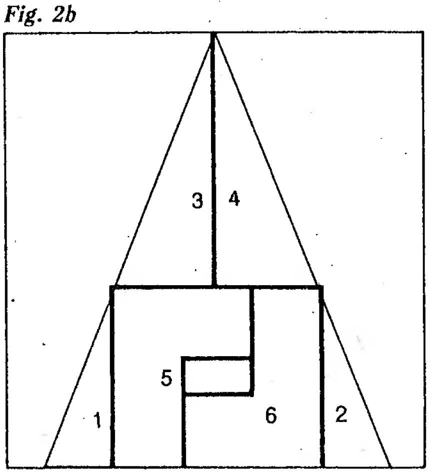
Fig. 3

these pieces from graph paper and try them before turning to page 76 for solutions.)

These are typical examples in recreational mathematics which use geometrical aspects to full advantage. Probably geometrical recreations are most striking because one expects geometry to be the epitome of logic and reason. Besides, geometrical constructs have an easy flexibility which allows them to merge smoothly with other branches of mathematics as we shall see.

The simplest kind of geometrical teaser involves "fallacies"; the earlier examples actually belong to this category. A standard fallacy involves a series of apparently correct reasonings, leading to a manifestly absurd conclusion. The most famous among these "proves" that all "triangles are equilateral". A minor variant of this fallacy "proves" that "any quadrilateral ABCD which has an angle A equal to angle C and AB equal to CD is a parallelogram." (A parallelogram is a four-sided figure with both pairs of opposite sides equal).

Fig. 2b



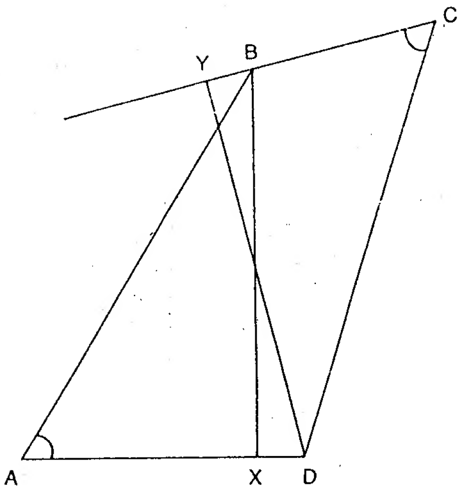


Fig. 4

The reasoning is based on Fig. 3. "In the quadrilateral ABCD,draw BX perpendicular to AD and DY perpendicular to BC and join BD. Triangles ABX and CYD are congruent making BX equal to DY and AX equal to CY. This, in turn, makes triangles BXD and DYB congruent, and so XD equals YB. Thus AB equals CD and AD equals BC and ABCD must be a parallelogram."

The reasoning happens to be perfectly correct and yet the conclusion is false! The fallacy is in the figure. A quadrilateral ABCD is shown in Fig. 4, which meets the requirements of the theorem (that is, AB equals CD and angle A equals angle C), but is nevertheless not a parallelogram. As the reader can see, X is between A and D while Y lies on a line which is an extension of BC. The reasoning given above has not taken this eventuality into account! (This fallacy was discussed in *Mathematical Gazette*, Oct. 1959, p. 204).

Fallacies, however, are rather easy to crack because one can always start with a counter-example (that is, a situation in which the claim is not true) and work backwards. More ingenious problems do not offer any clear-cut method of attack. For example, consider the following question: given any simply connected figure, is there a straight line that simultaneously divides both the area and the perimeter into equal halves? The answer, which is far from obvious, happens to be "yes" and one possible proof is shown in Fig. 5.

Choose two points P and Q which bisect the perimeter of the figure, and draw a line through PQ. If the shaded and unshaded areas are equal, we have already achieved what we want. Otherwise proceed as follows: shift P, Q along the curve continuously, always maintaining the bisection of the

curve. Finally, when the line PQ has rotated through 180° (P&Q would have interchanged the positions), the shaded area would have become unshaded and vice versa. The arrow in the figure will now be pointing upwards. Consider the difference in area between the shaded and unshaded regions. Originally it was, say, positive, that is, the shaded area was larger than the unshaded area; at the end of the 180° rotation, it is negative. Since the change was continuous, there must have been a position in which the area difference was zero, or the areas were equal. At that position, PQ bisects both the area and the perimeter.

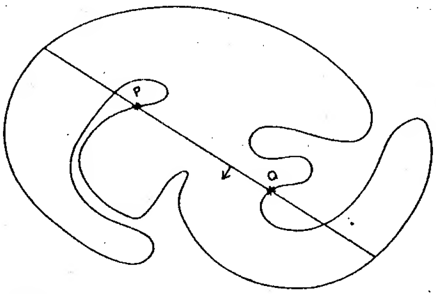


Fig. 5

Clearly, the above problem requires a little bit more than purely "geometrical" reasoning. Often the extra bit of analysis will be based on an entirely non-geometrical concept, making the question harder to answer. As a second example, consider the following problem: "Consider a circular region of 16 cm radius, and a circular ring of outer radius 3 cm and inner radius 2 cm. A random set of 650 points are placed on the circular region. Show that irrespective of the arrangement of points, the ring can now always be placed covering at least 10 points."

If a successful line of attack suggests itself to the reader on such an esoteric problem, he must be congratulated. The simplest reasoning that the author knows, goes as follows. Imagine for a moment that you are provided with 650 rings, rather than one ring. Place them on the circular region with each of the 650 points as a centre. Some of the rings might project outside the original circle of 16 cm radius, but they will all be within a circle of radius 16 + 3 = 19 cm. We shall call this the outer circle. The first step in the reasoning is to note that there must be at least one point which is covered by at least ten rings. How do we know this? Well, suppose each point is covered *at the most* only by nine rings. Then the total area of all rings must be less than nine times the area of the outer circle which, however, is not the case (the area of 650 rings is $650\pi\ (3^2 - 2^2)$, that is, 3250π, while nine times the area of the outer circle is only $9 \times 361\pi$, which is 3249π). Thus there must be at least one point (call it X) which falls under ten rings.

It is now easy to arrive at the necessary conclusion. Let $Y_1, Y_2, \dots Y_{10}$ be the centres of the ten rings that cover X. Clearly, a ring kept at X will now cover at least the ten points $Y_1, Y_2 \dots Y_{10}$!

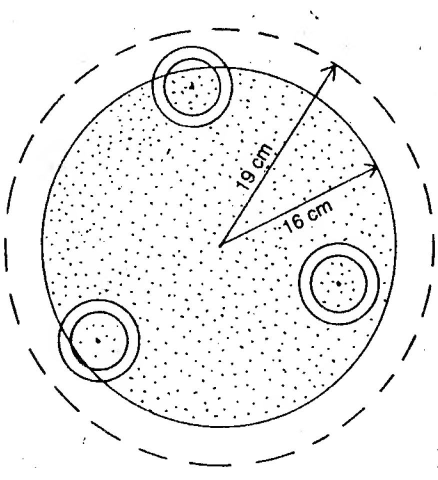
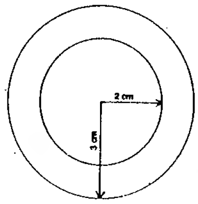


Fig. 6



— MUKUND TALWALKAR

The above reasoning (due to V. Linis of the University of Ottawa, Canada) illustrates many aspects of non-trivial problem-solving: (i) imagination to consider 650 replicas of the ring, (ii) application of "pigeon-hole principle" to prove the first step, and (iii) "inverting" the one-point-covered-by-ten-rings situation to one-ring-covering-ten-points situation. A much simpler problem along similar lines is given at the end of the article for the reader to attempt.

The following question, however, demands entirely different kind of imagination. In Fig. 7, two regular hexagons are drawn, one inside a circle and the other outside. If the inside hexagon has an area of 3 square cm, what is the area of the outside

An eye for *figures?*

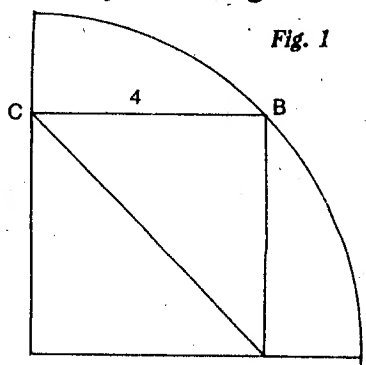


Fig. 1

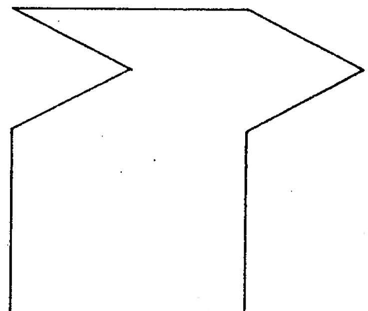


Fig. 2

THESE two 'quickie' problems can be a good test of your geometrical aptitude: (1) In Fig. 1, in this box, what is the length of AC? (2) Cut Fig. 2 into four congruent pieces.

And do not give yourself any credit if you take more than one minute to solve these two problems.

(See page 76)

Please send your solutions to the problems discussed in this article to SCIENCE TODAY

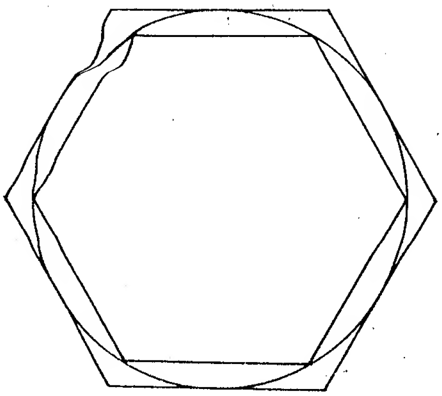


Fig. 7

hexagon? From the inside hexagon's area, one can calculate the radius of the circle and then compute the side of the outside hexagon and, at last, the area. That, however, is a wrong approach to problem-solving, besides being laborious!. With some geometrical imagination, one can get the answer without doing anything but a little mental calculation (the reader can find the answer on page **76**

We conclude with three problems for the reader to try out. Please send your answers to SCIENCE TODAY. (The answers will be discussed in a later issue.)

(1) Five points are placed inside an equilateral triangle of side 1 cm. Is it possible for each pair among them to be separated by more than 0.5 cm?

(2) Fig. 8 shows three squares kept adjacent to each other. Show, without

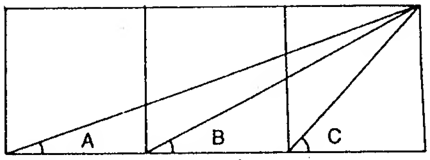


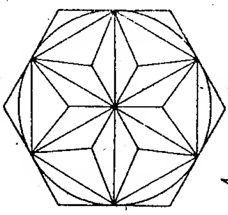
Fig. 8

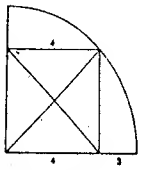
using trigonometry, that A + B = C. (There are many ways of proving this; the idea, of course, is to choose the simplest way!)

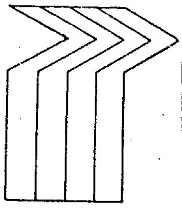
(3) There is a point P inside an equilateral triangle ABC such that PA, PB, and PC are 3, 4 and 5 cm. What is the side of the triangle?

T. Padmanabhan

Dr. Padmanabhan is with the Astrophysics Group at the Tata Institute of Fundamental Research, Bombay. He has a keen interest in recreational mathematics.







PLAY THEMES

FIGURE IT OUT

See page 54

IN Fig. 1(b), (page 54), the red diagonal looks like a nice straight line; but it is not! This is because the slopes of the meeting edges of pieces numbered 4 and 2 are different. So the "reactangle" is not really a closed rectangle. (Those who know a bit of trigonometry can easily calculate the difference in slopes to be $\tan^{-1}1/46$ which is about 1.25°).

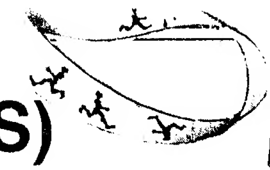
It is the same story with figure 2(b). The longest sides of triangles 2 and 4 are not in a straight line. In other words, Fig. 2(b) is not a triangle at all.

The hexagon problem involves "rotating" the hexagons to the position shown in figure (A) here. Since the small triangles are congruent, the area of the outer hexagon is 3×24/18 sq cm, that is, 4 sq cm.

The answers to 'quickie' problems may be obvious from figures(B)and(C).

PLAY THEMES

TALKING CHANCES (SOLUTIONS)



IN the June issue, "Talking Chances" (Play Themes) carried three problems for the readers to answer. We shall discuss the solutions in this issue. Comments, alternative solutions (and criticism) are welcome and will be discussed in a future issue.

The first problem was based on simple word play. James thinks "Either I win or lose..." and proceeds on the assumption that it is *equally probable* for him to win or lose. This clearly need not be the case. For example, assume that the combined total monetary wealth available to James and Ramesh is Rs.100,000. Then if James is carrying with him more than Rs.50,000 he is sure to lose the bet. Thus in any practical situation, the man carrying the larger sum of money has a greater chance of losing it.

The answer to the problem of red ace is $\frac{1}{3}$. Whatever is the distribution of cards, there is bound to be one black ace with either B or C. So the fact that D could produce a black ace *after peeping* into the hands of B and C does not add any further information regarding A's card.

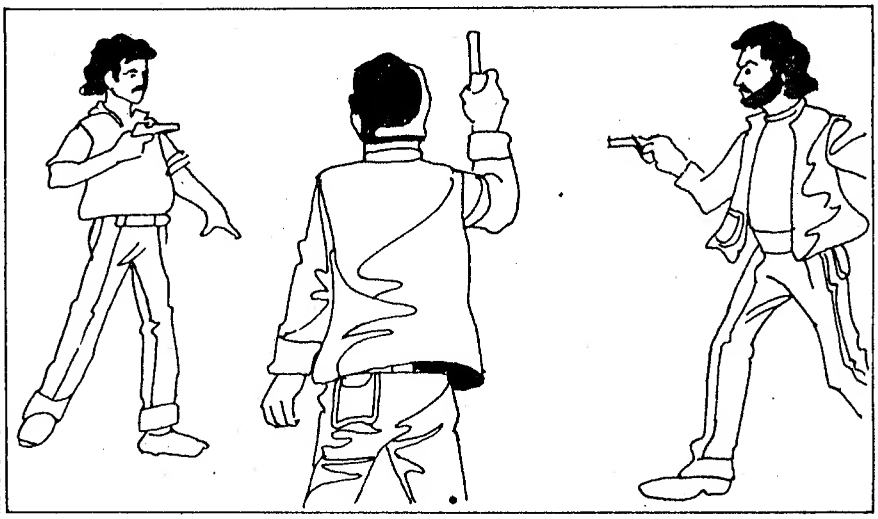
The situation can be made more dramatic. Suppose a deck of cards is shuffled and a single, randomly chosen card is dealt to A, what is the chance that the card is ace of spades? Clearly, it is $\frac{1}{52}$. Now suppose someone looks through the remaining 51 cards in the deck and turns over 50 cards none of which is ace of spades. What is the chance that the card A is holding is ace of spades? Is it $\frac{1}{2}$ or is it still $\frac{1}{52}$? Though it may be startling at first, the answer is still $\frac{1}{52}$. Whatever is the card that is dealt to A, someone can always produce 50 cards out of the remaining 51 which are *not* ace of spade. Therefore this gimmick does not give any further information which can change the odds.

The third problem, probably, was the toughest. It turns out that the poorest shot, C, has the highest survival probability! If everybody adopts the best strategy then C has an overall survival chance of $\frac{47}{90}$. A has the next best of $\frac{3}{10}$ and B has the lowest of $\frac{8}{45}$.

To compute these probabilities, one should first decide the best strategy for each. The best strategy for A or B, of course, is to aim at each other in their turns so as to eliminate the most dangerous opponent. But what is the plan that C must adopt? He should shoot in the air (or into the ground!) until A or B is dead. This gives him a first shot advantage at the survivor, boosting his chances.

The survival chances of A are the easiest to determine. He has a 50 per cent chance of getting a first hit at B, thereby killing him. B also has a 50 per cent chance of getting the first shot, in which case A has a

$\frac{1}{5}$ (remember that B is 80 per cent accurate) chance of surviving him. Once he survives, of course, he shoots down B in his turn. Thus the total survival chance of A against B is $\frac{1}{2}$ added to $\frac{1}{2} \times \frac{1}{5}$, which is $\frac{3}{5}$. So far we do not have to worry about C because he has been shooting in the air; but the moment A kills B, C gets a chance to fire and he aims for A. Since C is a 50 per cent shot, A has a $\frac{1}{2}$ chance of surviving. (Once he survives, of course, he shoots and kills C). Thus the overall survival chance of A in the dual is the product of his chance against B ($\frac{3}{5}$) and his chance against C ($\frac{1}{2}$), that is, $\frac{3}{10}$.



MUKUND TALWALKAR

The survival chance for B is somewhat more complicated because we get into an infinite cycle. To begin with, his survival chance against A is $\frac{2}{5}$. (We saw earlier that A's survival chance is $\frac{3}{5}$; so B's must be $\frac{2}{5}$). The moment B has survived, shooting down A, he faces fire from C. There is a $\frac{1}{2}$ chance that C will miss, in which case B has a $\frac{4}{5}$th chance of killing C. Thus upto the end of the "first round", B's chance of getting out in one piece is $\frac{1}{2} \times \frac{4}{5}$, or $\frac{4}{10}$. On the other hand, there is a $\frac{1}{5}$ chance of B missing which will allow C to have another crack at him. Thus the chance of B surviving the 'second round' is $\frac{1}{2} \times \frac{1}{5} \times \frac{1}{2} \times \frac{4}{5} = \frac{4}{100}$. Clearly, this process goes on ad infinitum. The overall survival chance of B is, therefore, given by the infinite sum ($\frac{4}{10} + \frac{4}{100} + \frac{4}{1000}$...), that is, 0.4 + 0.04 + 0.004, giving 0.444 ... The repeating decimal 0.444 .... is same as $\frac{4}{9}$ which is B's survival chance against C. Earlier we saw that B has a survival chance of $\frac{2}{5}$ against A. Thus the overall survivability of B in the duel is $\frac{2}{5} \times \frac{4}{9}$, or $\frac{8}{45}$.

A similar reasoning will give the survival chance of C as $\frac{47}{90}$. Of course, it is also obtained by substracting the sum of A's chance ($\frac{3}{10}$) and B's chance ($\frac{8}{45}$) from 1.

This problem is an old chestnut and has appeared in various puzzle books in different forms (a detailed calculation appears in *American Mathematical Monthly*, December 1948, p.640). The most surprising feature is that the poorest shot (C) has the highest survivability, when he keeps clear of "superpower rivalry". Martin Gardner, while discussing this problem in his column mentioned that "there is a moral of international politics in it".

However, suppose C is not so clever and, against his best interests, shoots at A or B in his turn. What are the chances in this case? A detailed analysis shows that C still has the best chance of survival (44.722 per cent), while B has the second best chance of 31.111 per cent. The sure shot, A, has only a survival chance of 24.167 per cent. Clearly, a foolish "neutral" country can be a big handicap to a superpower.

Several readers had sent their solutions. Among them, the best were from M. S. Srikanth of Madras who had tackled all the problems correctly. A. Mathew of Trichur has also sent detailed analysis of the problems, but has assumed that C shoots at A rather than in the air. As explained above, this changes the chances somewhat.

T. Padmanabhan

Dr. Padmanabhan is with the Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

PLAY THEMES

IT'S ALL IN THE CARDS

"Do you like card tricks?"
"No, I hate card tricks," I answered.
"Well, I'll just show you this one".
He showed me three.
—"Mr. Know. All", Somerset Maugham

FROM the standard deck of 52 cards remove the red kings, black aces, red fours, six of clubs and jack of diamonds. Shuffle the deck, hold it face down and take the cards from the top in pairs. If the first pair contains one red and one black, throw them away. If the pair is red, keep it at some place on the table, starting a red pile. If it is black, put it separately, forming a black pile. Run through the deck thus discarding all the red-black pairs and forming two separate piles of red and black cards. Count the number of cards in the red pile as well as the black pile and substract the smaller number from the larger. What is the most probable answer?

The reader should probably decide on an answer before reading further. Since the deck is well shuffled, the distribution of colours in pairs, as they are chosen, is entirely random. The number of cards in the red pile and in the black pile will be different at each trial. But the incredible fact is that there will always be two more cards in the black pile than in the red one! Notice that we had removed five red cards and three black cards from the deck of 52, leaving 21 red cards and 23 black cards. Suppose n pairs of red-black are discarded. This clearly leaves (21-n) red cards and (23-n) black cards, that is, exactly two more black cards than red ones!

The above colossal swindle is used as the basis of many magic tricks. The simplicity of the principle in contrast to the final effect is characteristic of the numerous recreational problems involving cards. Mathematical recreations involving playing cards can be classified into three broad types: (i) arrangements, (ii) probability estimates and gamblings, and (iii) games of strategy. Let us look at some examples of each kind.

Here is a simple problem involving card arrangement. Take three aces from a pack and keep them face down on the table. The task is to turn one card at a time and (in seven moves) produce all the 2x2x2=8 different permutations of face-up and face-down cards. It turns out that there are six ways of doing it. Let F denote a face and B denote a back. Then one possible solution is BBB, BBF, BFF, BFB, FFB, FBB, FBF and FFF. Can the reader now do it with all the *four* aces kept down on the table? That is, starting with four face-down cards and turning one at a time, one should run through all the 2x2x2x2=16 permutations in the next 15 moves, ending up with all the

four aces face up (solution is given in page 74, 1).

The existence of four 'court cards' (ace, king, queen and jack) in four suits offers an interesting 'magic square' problem. Arrange the 16 court cards in the form of a 4x4 square such that no row, column or diagonal contains the same face value or suit more than once. It is a reasonably simple problem to solve, provided the reader does not approach it in a blind trial and error fashion (see p. 74).

A somewhat different kind of logic is required when the conditions of arrangement are specified without revealing the cards. And this one is remarkably cute: Three cards, removed from a standard deck, lie face down in a horizontal row; to the right of a king there is a queen or two. To the left of a queen, there is a queen or two. To the left of a heart, there is a spade or two. To the right of a spade, there is a spade or two ('two means two cards, of course, not two of a suit!). What are the three cards? (p. 74, 3).

Here is a last example based on card arrangements. Take a good look at Fig.1. You are told that all the backsides of the cards are either coloured or black. Suppose you are asked to answer the question, "Are all the cards with coloured backs jokers?". What is the *minimum* number of cards that must be turned over in order to answer this question? The first, impulsive answer, will be "four". After some reasoning, one will come up with the answer "two". I must confess that there is a bit of "cheating" in this problem and the answer "two" is not correct! The reader should think again, before turning to page 74, 4.

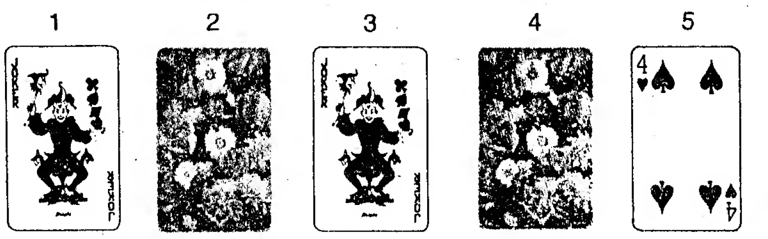
Many gambling tricks are based on probability calculations regarding playing cards. A very old chestnut which has appeared in different forms is the following. A gambler offers to the victim the following proposition at equal odds; the gambler and the victims take one pack of cards each and shuffle it well. They simultaneously deal the top cards of their respective packs to the table, face up. If the cards are the same (that is, a perfect match), the gambler wins; if not, they deal the next pair to the table and look for a match. If no match is found until they exhaust the whole pack then the victim wins. Superficially, it might look like an attractive bet 'favouring' the victim at equal odds. Most people would think the chance of coincidence to be low, certainly less than ½. The probability is actually 0.63! In other words, it is more likely for a coincidence to occur than not.

The problem is usually stated in a different manner. Suppose you have written N letters to N people and randomly insert the letters into N addressed envelopes. What is the probability that *none* of your friends will get the right letter? For N=52, this is the same as the probability for no coincidence in the card game. We can calculate this probability as follows:

Let X_N denote the number of ways all the N letters can be misplaced. Suppose the first letter is in a*th* envelope and b*th* letter is in the first envelope. Either a=b or a ≠ b. If a = b, there remain (N-2) letters that have to be placed wrongly, which can be done in X_{N-2} ways. Since a (=b) can take any value from 2 to N, this possibility covers (N-1) X_{N-2} cases. Now consider a ≠ b situation. For the moment, fix b and let a run over 2 to N except b. In the (N-1) envelopes other than the first, we have to place (N−1) leters other than b*th*, but the first letter must not be kept in the b*th* envelope. Some thought shows that this is again X_{N-1}. Now letting b vary, we cover (N-1) X_{N-1} cases. Thus the total number X_N must be the sum of cases with a=b [which is (N-1) X_{N-2}] and cases with a ≠ b [which is (N-1) X_{N-1}]. Thus we arrive at the relation $X_N = (N-1)(X_{N-1} + X_{N-2})$. Using $X_1=0$ (single letter, no error) and $X_2=1$ (two letters, only one wrong way), we can rapidly compute X_3 (=2), X_4 (=9), X_5 (=44), etc. The general formula turns out to be $X_N = N!\,(1-1/1! + 1/2! - 1/3! ... \pm 1/N!)$ (The exclamation mark after an integer stands for the "factorial". N! is short hand for (1x2x3x...N). For example, 2! = 2×1 =2; 3! = 3×2×1 = 6; 4! = 4×3×2×1 = 24 etc).

Since N letters can be put in N envelopes in altogether N! ways, the probability for

Fig. 1



SO YOU THINK YOU'RE NOT BLIND, EH?

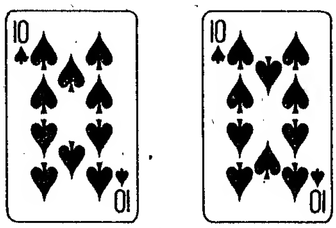
IT is surprising how blind one can be to details. Test your 'blindness-quotient' by trying out this quiz. All questions are based on a "standard" pack of 52 cards.

1. How many eyes are depicted in the pack?
2. How many moustaches are depicted?
3. In the full suit of spades, how many times does the spade symbol appear?
4. In Fig. (3), which is the "correct" ten—(a) or (b)?
5. Which King does not have any sword (or dagger)?
6. Again, which is the "corect" four, Fig. 4(a) or (b)?
7. If all the numbers that appear in the 13 cards of suit of spades are added up, what will you get?
8. In picture cards, some show only half the face. Others which show most of the face depict the face as turned to left or right (slightly). Question: How many are turned to left and how many to right?
9. What weapon does King of diamond have?
10. And a simple question: what does Queen of hearts have in her hand?

(See p. 74 for answers.)

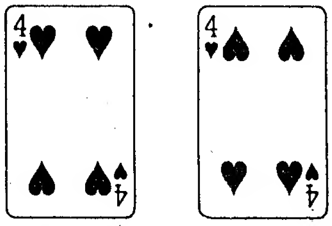
T.P.

Fig. 3



(a) (b)

Fig. 4



(a) (b)

"all wrong" is just $(X_N/N!)$ which is $(1-1/1! +1/2! \ldots 1/N!)$. This number is about 0.36 for N = 6 and higher! (Larger and larger values only change higher decimal places; 0.36 remains the same.) Thus with N = 52 cards the chances of the victim winning is about 36 per cent! (Yet another problem involving rather unlikely probabilities is given in the end of the article for the reader to solve).

Let us now turn to a card game that involves certain mathematical principles. One simple game involves just five cards (ace, 2, 3, 4, 5 of any suit). The first player arranges the five cards in any particular order and keeps them face down. The second player now picks up any pair of *adjacent* cards. If one of the picked-up cards is an ace, the second player ("picker") gets a point; if it is not an ace, the first player ("placer") is the winner and nobody gets any point. A sequence of games is played with the players alternating the roles. One who emerges with higher points in the end is the winner.

This apparently simple situation can easily drive one mad. Suppose you are the placer where should you keep the ace to achieve maximum advantage? (Fig.2). If you keep the ace in one of the two extreme positions (A or E), you seem to have an advantage. Any middle position B, C or D will get picked up in two choices of the picker. (For example, suppose you place the ace at position B, you lose for both choice 1 (A and B) and choice 2 (B and C). The extreme position A, for example, will be picked up only in a single choice (choice 1) of the picker. So, probably you should keep the ace in A or E. But wait a minute! Your friend will be reasoning out the same way and will guess that you are going to keep it at A or E, and will make choice 1 (or choice 4). So it is probably better to mislead your friend by actually keeping the ace in B, C or D. But then, he may be cleverer and expect you to do just that! We have walked into a real labrynth of circuitous reasoning.

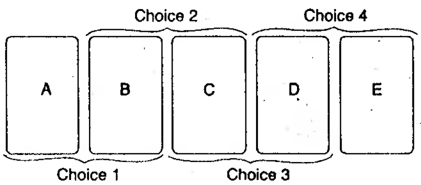
Isn't there some systematic way of analysing this problem? Game theory provides one such way called 'mixed strategy'. The idea is to place the ace in A, C or E, selecting the position randomly with probability 1/3 for each game. (To make sure that the choice is truly random, roll a dice and place the ace at A if the dice comes up 1, 2; at C for 3, 4; and at E for 5, 6.) In other words, the placer should randomly mix his strategies ("mixed strategy") of middle play and extreme play. (The reader can easily calculate the probability that this strategy will fool the picker. It is 2/3.) What is the best strategy for the picker? He should choose 1 and 4 with probability 1/3 each and 2 and 3 with probability 1/6 each. (He can roll a dice and choose 1 if 1 or 2 turns up; choose 4 if 3 or 4 turns up; choose 2 if 5 turns up and choose 3 if 6 turns up). This ensures him the best chance for success which 1/3.

This simple game is quite interesting to play out because (as long as you don't follow the above strategy) conscious biases always creep in. Very soon, one of the players will be able to outguess the other, and win a string of games.

Here are some problems for the reader to tackle. Send the answers to SCIENCE TODAY.

1. Remove the spades and hearts from a pack. Place the spades in a row (face up) serially (ace, 2,3...K). Place a heart card under each one of these so that the sum of the two cards is a perfect square. (For example, you are allowed to keep 3H or 8H

Fig. 2



below 1S, bringing the total to $3+1=4=2^2$ or $8+1=9=3^2$. Treat aces as 1, Jacks as 11, Queens as 12 and Kings as 13).

2. A standard deck of 52 cards is shuffled and placed face down. What is the most probable position from the top (first, second, third...fiftyth etc) for the first black ace?

3. Take from a pack of cards the following cards: two aces, two twos, two threes, two fours. Arrange them in a row so that one card separates the two aces, two cards separate the twos, three cards separate the threes and four cards separate the fours.

A reader looking for a really challenging problem can attempt the following generalization of the above question. Take from a deck 27 cards bearing values 1 to 9 of spades, hearts and diamonds. Arrange these 27 cards so that between the first two cards of value k there are exactly k cards and between the second pair with value k again there are k cards. For instance, there must be five cards between the first two 5's and again five cards between the second and third 5's; 5 ★★★★★ 5 ★★★★★ 5 and so on.) While the original problem is easy, this generalization is quite tedious and difficult.

4. Take the cards ace, 2... 10 of suit, shuffle and arrange them in any order you want. No matter how it is arranged, it will always be possible to pick out *at least* four cards (of course, there can be more) in ascending or descending order. Prove the above statement! For instance, suppose the arrangement is 5, 7, 9, 2, 1, 4, 10, 3, 8, 6. We see that 5, 7, 9, 10—four cards—are in ascending order. Can we beat the theorem by, say, moving 10 between 7 and 9? No, because this creates a 10, 9, 8, 6 descending order!

5. Here is a problem of practical importance! You are dealing the cards in a game of bridge, say. Part of the way through the deal, a telephone call interrupts you. When you return to the table, neither you nor your friends remember where the last card was dealt. Without doing *any counting*, how can you continue to deal accurately, everyone getting exactly the same cards he would have had if the deal was not interrupted?

T. Padmanabhan

Dr. Padmanabhan is with the Astrophysics Group at the Tata Institute of Fundamental Research, Bombay.

More solutions — and some problems

HIS month, we will discuss solutions to the two earlier problems—"Games(wo)manship" given in the July issue and "Figure it Out" (August). We will also discuss some new problems in the end. Incidentally, a large number of readers have sent solutions to the problems in "Figure it Out".

The first two problems in "Games (wo)manship" are actually about the games of Tick-tack-toe and Nim presented in disguise.

The first game of picking up cards from nine cards will end in a draw if both players play rationally. To see this, imagine that the nine cards are arranged in the form of the "magic square" shown in Fig. 1. Notice that the sum along rows, columns and diagonals, all lead to 15. The game described in the problem is thus completely equivalent to the familiar game of Tick-tack-toe (or

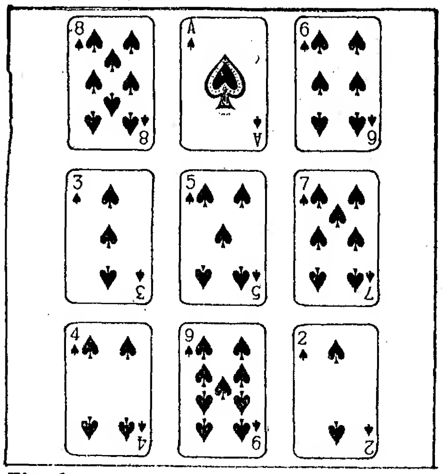
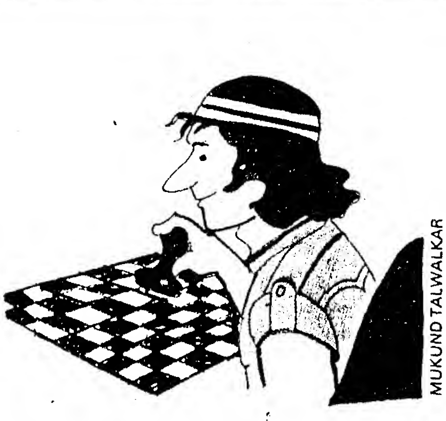


Fig. 1

"crosses and noughts" or "three-in-a-row") discussed in the article! A player picking up a card is equivalent to his making a mark at the appropriate cell; his attempt to secure a total of 15 is the same as an attempt to get three marks in a row. Having reduced the problem (mentally, of course) to Tick-tack-toe, it is trivial to work out the strategy. If both players proceed rationally, the game ends in a draw as in Tick-tack-toe.

The second game is a variant of 'Nim' which was described in the box on p. 61 of the same issue. Let us first count the number of vacant cells *between* the white and black pawns in each vertical row of the chessboard (see Fig. 6 in the July issue). From left to right, the number of vacant cells are 4, 3, 4, 2, 2, 5, 4 and 2. Everytime a player makes a move, he is reducing the number of vacant cells in one of the rows. (For example, if white moves the extreme left pawn up by 3 squares, he will change the configuration to 1, 3, 4, 2, 2, 5, 4 and 2). Thus the game is logically identical to the game of Nim with eight heaps of counters,

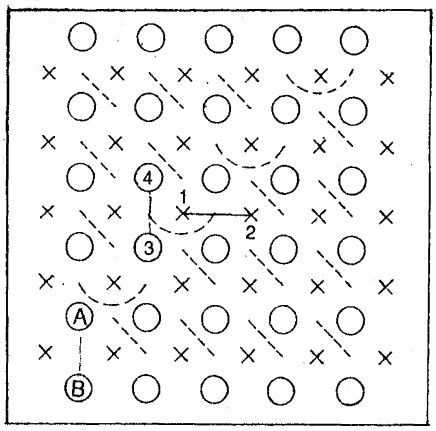


with the first heap containing 4, second 3, third 4, etc (corresponding to the initial number of vacant cells). Each move is equivalent to removing counters from some heap or the other. Once all the vacant cells have vanished, the configuration reduces to 0,0,0,0,0,0,0,0, indicating a position with all pawns locked up. The player who made the last move has clearly "picked up the last counter", in the equivalent game of Nim.

The rules given in the box (p.61, July 85) for analysing the game of Nim can now be used. Writing the initial position as the sum of powers of two and removing pairs (4 = $\cancel{4}$, 3 = $\cancel{2}$ + $\cancel{1}$, 4 = $\cancel{4}$, 2 = $\cancel{2}$, 2 = $\cancel{2}$, 5 = $\cancel{4}$+$\cancel{1}$, 4=$\cancel{4}$, 2=$\cancel{2}$), we find that the position is 'safe'. Any move the first player makes will bring it to an unsafe position for him. The second player (if he is careful!) will reduce it back to a safe position. In other words, the second player can win in this game, by converting "unsafe" to "safe" positions in each turn.

The last problem—probably the toughest—is to work out the winning strategy for the game of 'Bridge—It'. Many strategies exist, but the most elegant one has come from Oliver Gross of Rand Corporation, USA. The key to this strategy is shown in Fig. 2. Make the first move as indicated by the coloured line between A and B. Then whenever your opponent connects two points by a line, you are required to connect

Fig. 2



the two points "paired to" the opponent's choice by the dotted lines in the figure. An example will make this rule clear. Suppose your opponent connects points 1 and 2 (Fig. 2). We see from the figure that there is a dotted line between (1,2) and (3,4). Therefore, you make your move connecting (3,4). The dotted lines in the figure show the reply to any choice made by your opponent. (Of course, to win in a practical game, you must memorise the dotted line pattern shown in Fig. 2.) There are no dotted lines connected to the extreme border of the board. If your opponent makes a move connecting two border points (merely to confuse you, since border play gives no advantage), then you can do the same. The reader can easily convince himself that the method works by playing out some sample games. A striking feature of the above strategy is that it plays intelligently against

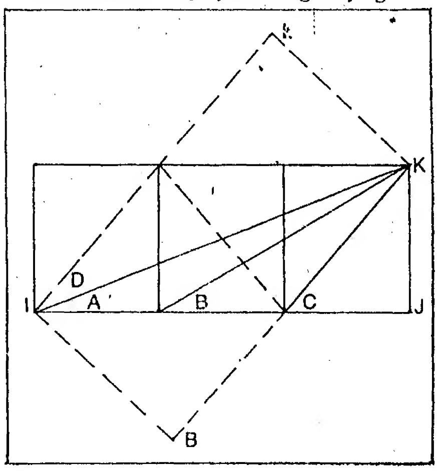


Fig. 3

an intelligent player and stupidly against a stupid player, but always keeping one step ahead.

Geometrical problems

The first of the geometrical problems in the August issue (p.54) can be solved by using the properties of similar triangles (see Fig. 3 here). We constructed two more squares as indicated by dotted lines. Since IL = 2KL and BJ = 2KJ, triangles IKL and BJK are similar, making angle D equal to angle B. Thus A + B = A + D which is same as angle C. Of course, there are many other ways of proving the result. Correct but more elaborate solutions to the above problem were sent by Mr. G. V. Khare, Mr. R. Kothandaraman, Mr. M. Kannan and Mr. C. G. Subramaniam.

To solve the second problem, we only have to divide the equilateral triangle into four smaller triangles as shown in Fig. 4. When five points are chosen inside the original triangle at least one small triangle must get more than one point! Since the

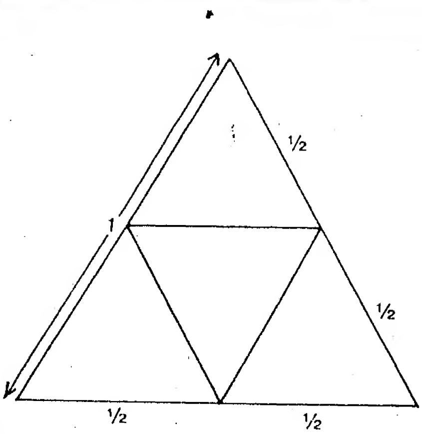


Fig. 4

side of smaller triangles is ½ unit, two points within the same small triangle cannot be farther than ½ unit. All the readers who submitted the solutions have solved this problem!

The last problem is to determine the side of an equilateral triangle which contains a point inside, which is at distances 3, 4 and 5 units from the vertices (see Fig. 5). We have constructed an equilateral triangle PCF with side three units and drawn AE perpendicular to CP produced.

We begin by noticing that triangles ACF and PCB are congruent [AC = BC; FC = PC; angle ACF = (60° – angle PCA) = angle PCB] and hence AF is 5 units. Since triangle APF has now sides 3,4 and 5, it is a right-angled triangle, making angle APF = 90°. This means that in triangle AEP, angle EPA is 30° and angle EAP is 60°. Thus, the side AE must be half of side AP, namely 2 units, and EP must be the square root of $(4^2 - 2^2)$ which is $2\sqrt{3}$. We now know two sides EC ($=EP+PC = 2\sqrt{3} + 3$) and AE ($= 2$) of the right-angled triangle AEC. It is a trivial matter to compute the hypotenuse $AC = \sqrt{2^2+(2\sqrt{3} + 3)^2} = \sqrt{25 + 12\sqrt{3}}$ which is approximately 6.76. Thus the side of the equilateral triangle is 6.76. (This problem is a special case of a more general situation described in *Ingenious Mathematical Problems and Methods* by L. A. Graham, Dover, 1959, problem 55.) Mr. G. V. Khare, Mr. K. Arjan, Mr. Kothandaraman and Mr. M. Kannan have sent correct solutions which are based on similar analysis.

We conclude with an assorted collection of problems for the reader to tackle. Send the answers to SCIENCE TODAY.

1. You are given a common balance and six balls. One pair of balls is red, one pair white and one black. Within each pair one ball is a bit heavier than the other. The three heavier weights (one of each colour) all weigh the same. Similarly all the three

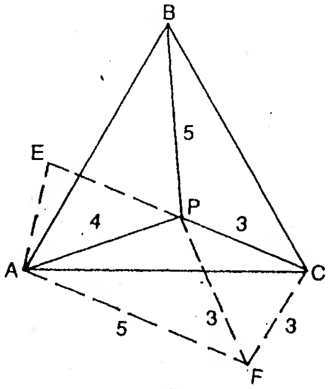


Fig. 5

lighter ones weigh the same.

It is quite easy to identify the heavier weights in *three* weighings. Simply put one red ball in each pan; the pan that goes down has the heavier weight. Repeat with black and white balls.

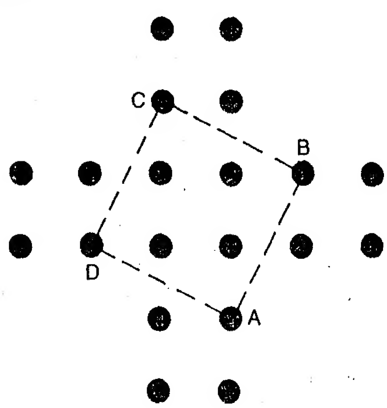
Now, can you identify the heavier weights in just *two* weighings?

2. Look at Fig. 6. Taking four counters to form vertices of a square, try to work the total number of squares in the figure. There are clearly five small squares in the horizontal row, two above the centre square and two below, making a total of nine. But there are many more squares in this figure which are not too obvious. For example, counters A,B, C,D mark the corners of a square which must be counted. Now, how many squares are there in the figure? We give one hint: there are altogether more than 20 squares.

3. Draw a circle of arbitrary radius. Put 10,000 points *randomly* inside this circle. Is it now possible to draw a straight line which (i) does not pass through any of the points, and (ii) divides the set of points into two halves so that there are 5,000 points on one side of the line and 5,000 on the other side?

Notice that the initial choice of the points is totally arbitrary. One could have chosen any random collection of 10,000 points.

Fig. 6



4. What is the missing number in the following set of numbers?

10,11,12,13,14,15,16,17,20,22,24,
31,100,—,10000.

(Hint: This is a difficult problem! Note that the set consists of 15 numbers.)

5. Find a ten-digit number which has the following property. The first (left most) digit indicates the total number of zeros in the entire number, the second digit from the left denotes the number of 1's in the number, and so on.

An examble of such a number with four digits (rather than with 10 digits as required above) is 1210. The first digit, 1, denotes the number of zeros, the second digit from the left, 2, denotes the number of 1's that appear in the entire number, etc. The problem is to produce a ten-digit number with the same property.

Fig. 7

6. Figure 7 shows a design for a desk calendar consisting of two cubes. The faces of each cube carries a single digit. The cubes can be arranged to get any date from 01,02... to 31. Can you determine the digits on the hidden sides of the cubes which will make this possible? It is somewhat trickier that one might think.

7. We conclude with a ridiculously simple question. In a town, a barber shaves *all* men who are not self-shavers, and does not shave any man who is a self-shaver. Now, is the barber a self-shaver or not?

T. Padmanabhan

Mr. Padmanabhan is with the Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

PLAY THEMES

All above the chess board

NEXT to cards, chess enjoys the maximum popularity among recreational mathematicians. The features we discuss here require absolutely no skill in chess; in fact, skill in chess can be a handicap at times! The reader, however, is expected to know *all* the rules of the game.

One of the simplest problems involving a chess board is illustrated in Fig. 1. This mutilated chess board has two diagonally opposite extreme white cells removed. Taking the side of each small cell to be one unit, the board shown in the figure has an area of (64−2)=62 square units. Now suppose you are given 31 pieces of small card boards each of which is a rectangle with dimensions (2 units x 1 unit). One such piece is shown by the side of the board in Fig. 1. Clearly, the area of the 31 pieces is the same (31 × 2=62) as the area of the mutilated chess board. Is it possible to arrange the 31 pieces on the board such that the board is completely covered?

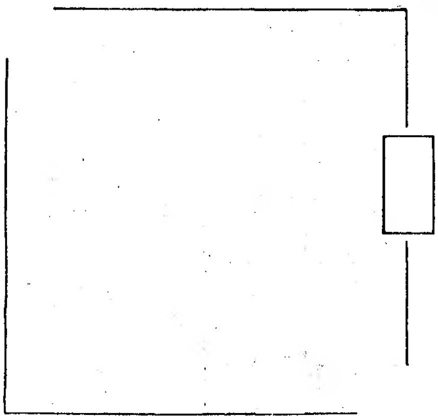


Fig. 1

A frontal attack on the problem, unfortunately, will lead to nowhere. Every arrangement which one tries, fails to meet the requirement. Can one really prove that the task is impossible? Nothing is easier if one goes about it in the proper way. Notice that every small rectangle, when placed on the board, must cover one black and one white cell. After, say, six cards are placed, 6 white and 6 black cells would have been covered; after 20 cards are placed, 20 black and 20 white cells would be covered and so on. Clearly we will only be able to cover equal number of black and white squares in the present arrangement. Our mutilated chess board, however, has 32 black and 30 white cells and hence cannot be covered by the small rectangles.

That raises the next question: suppose we remove from the chess board two arbitrary

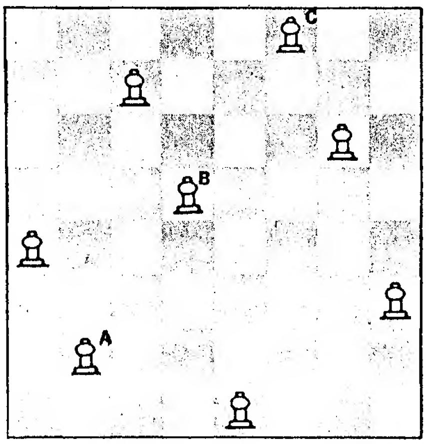
MUKUND TALWALKAR

cells of opposite colours—one white and one black. (The two cells may be *any* two on the board.) Is it now possible to cover the remaining region of the board with 31 rectangles? If the reader tries it out with any specific case, he will easily find that it can be done. The general proof that it can always be done is not difficult to obtain. Can the reader find it before it is discussed in a future issue?

Let us next consider some classic chess recreations which have acquired a vast lore of literature. The first one may be called the 'maximal queens' problems. How many queens can be placed in the standard chess board so that no two of them attack each other? Clearly, one cannot keep more than 8 queens. Simple trial and error will show that it is indeed possible to place eight queens in the standard chess board without two queens attacking each other. Figure 2 shows one such configuration.

A recreational mathematician is seldom satisfied with such simple answers; he asks further questions. Is the solution unique? If not, how many arrangements are possible? Is it always possible to keep non-attacking queens in an n × n "chess" board (in the standard case, of course, n=8)? All these problems have been analysed to great detail in literature.

Fig. 2



To begin with, the 8-queen solution in Fig. 2 is far from unique. There are 12 different ways of arranging the 8 queens on the chess board without two of them attacking each other. (Of course, one can obtain more configurations by rotations and reflections but they do not count as different.) It can also be proved that for any

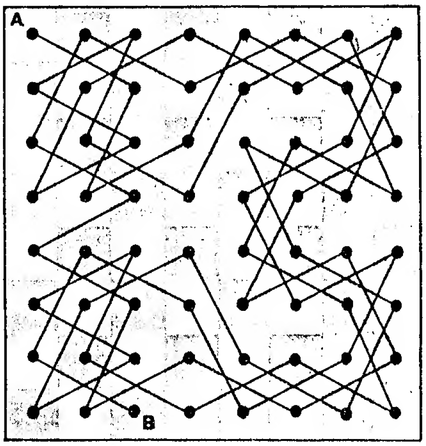


Fig. 3

1	30	47	52	5	28	43	54
48	51	2	29	44	53	6	27
31	46	49	4	25	8	55	42
50	3	32	45	56	41	26	7
33	62	15	20	9	24	39	58
16	19	34	61	40	57	10	23
63	14	17	36	21	12	59	38
18	35	64	13	60	37	22	11

Fig. 4

(nxn) board (with n>3) there is a solution with n queens on board. There is only one way of doing this in (4x4) board, two ways in (5x5), one way in (6x6) and six ways in (7x7). In a (9x9) board, there are 46 solutions while (10x10) has 92. (As far as I know, there is no general formula giving these numbers.)

If one looks at Fig. 2 configuration, one realises that there are three queens (marked A,B,C) which are in a straight line. (Of course, no two queens can be in the same vertical, horizontal or diagonal lines; but the problem did not impose any other condition.) Suppose we make the problem

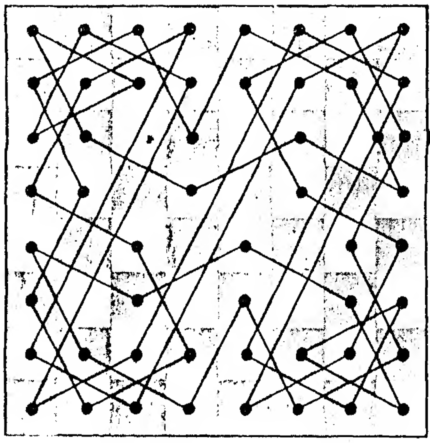


Fig. 5

tougher by changing it into the following form: place eight queens on a chess board such that (a) no two queens attack each other and (b) no three queens are in a straight line. Does the problem have solution?

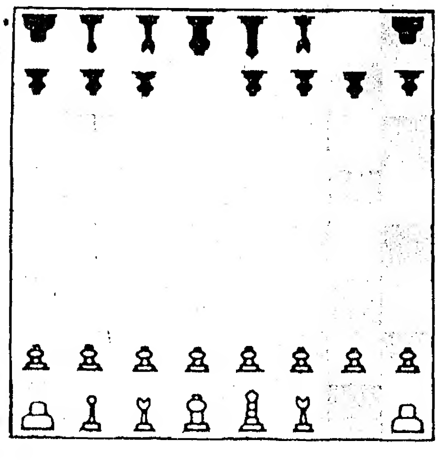
Yes! What is more, there is only one way (not counting rotations and reflections, of course) of placing the 8 queens in a chess board satisfying both (a) and (b) above! I will let the readers find it before revealing it in a future issue.

The problem of maximal queens can be generalised to other pieces in an obvious way; however, the answers are too simple to be of much interest. As an example, can the reader determine the maximum number of knights that can be placed on the chess board so that no two of them attack each other? (The answer is given on page 75)

The peculiar move made by knight makes it a special creature not only in chess games but also in chess puzzles. One of the "classic problems" dealing with the knight is called the "tour of the board". In its simplest form, the problem is the following. place a knight anywhere on a chess board. Now make a series of knight moves such that (a) the knight visits every cell in the board and (b) it does not land on any cell more than once.

Nobody knows how many such paths exist on the chess board. (The reader will be able to find one of his own with some patience!) Figure 3 gives an example of one such path starting at cell A and ending at cell B. In order to make the problem more restrictive, it is usual to impose some additional constraints. The solution in Fig. 3 has a really remarkable pattern hiden in it. To unveil it, we have shown in Fig. 4 the same solution numbering the knight's position consecutively (the knight starts at cell numbered 1, goes to 2, then to 3... and so on ending at cell numbered 64).The pattern is remarkable because the sum of the numbers along each row and each column add up to the same total of 260! Such an arrangement of numbers is called a semi-magic square. If the sum of the numbers along the diagonals also add up to the same total, the pattern would have become 'magic-square'. Nobody has yet succeded in constructing a magic square out of knight's tour. This challenging unsolved problem has driven many people mad! (Incredibly enough, such solutions are known in 16x16 board, for example!)

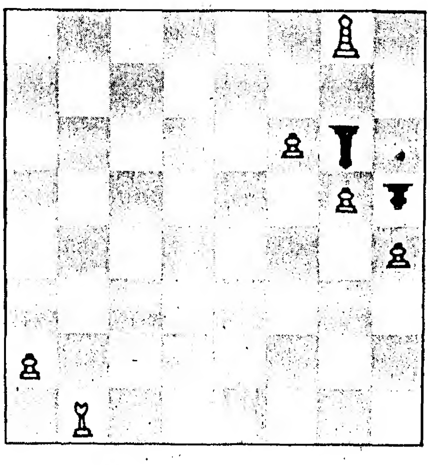
Fig. 6



A knight's tour is said to be "re-entrant tour" if it can jump from the final cell to the starting cell. One such example is shown in Fig. 5. One can start at any square and follow the lines touring the whole board. The beautiful geometric pattern in Fig. 5 is chracteristic of re-entrant tours: can the reader show that no re-entrant tour is possible in an (nxn) board if n is odd? (The answer is on page 75)For a more challenging problem, the reader can show that closed tours are impossible in any (4xn) board. The answer will be given in a future issue.

When more pieces are present on the board, the recreation possibilities are numerous. One particular class of these problems involves what is known as "retrograde analysis." In these problems, a particular chess position would be given and the reader will be led to the given position. It should be stressed that the moves are not required to be "good moves" but only legally valid moves. For example, consider the position in Fig. 6, which was reached after the fourth move. Can the reader reconstruct the first four moves of the game? (See page 75) If you were successful in this attempt, you may attack the following problems which require more of clear thinking than chess skill.

Fig. 7



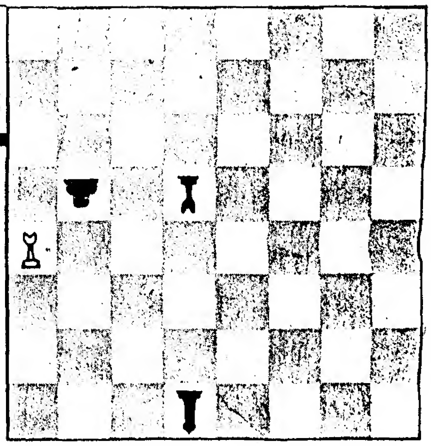


Fig. 8

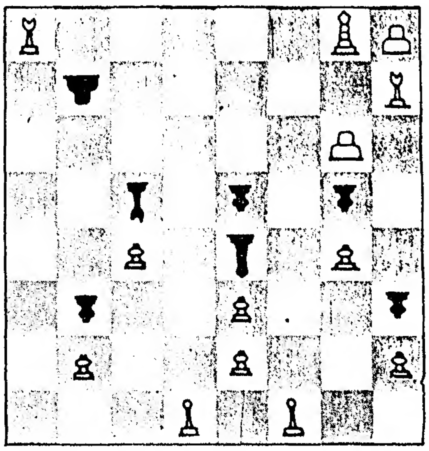


Fig. 9

1. See the position in Fig. 7. White has clearly checkmated black. But can the reader reconstruct the last two moves of black and white?
2. The position in Fig. 8 is incomplete because the white king is not shown. There is only one cell on the board where the white king could have been if this position can arise in an actual game. Can you find out where the king should have been and reconstruct the last few moves?
3. Normally, chess problems require "white to play and win." Well, Fig.9 is a problem with a difference. Find a move for white which will *not* result in an immediate win!

Send your answers to these questions and also to the three problems discussed in the text (covering chess board, 8 queens on chess board, re-entrant tour in 4xn board) to SCIENCE TODAY.

T. Padmanabhan

Dr. Padmanabhan is with the Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

PLAY THEMES

LOGICALLY SPEAKING...

'Contrariwise,' continued Tweedledee, 'if it was so, it might be; but if it were so, it would be: but as it isn't' it ain't. That's logic.

Lewis Carrol

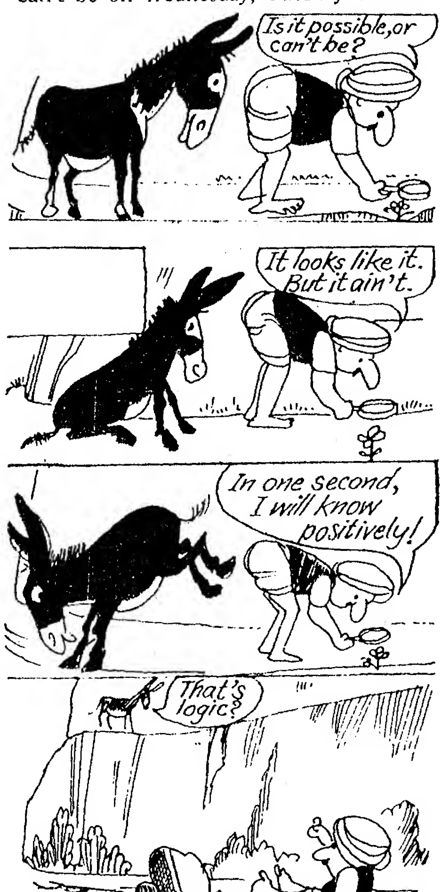
H come on now, be logical!' is a common expression in our everyday life. Most of us believe, essentially, in the power of logical reasoning. Given a set of assumptions, we expect ourselves to be capable of drawing logical conclusions.

Such a faith, most of the times, is well founded; but not always! There are many situations in which cold logic can lead to an unanswerable paradox. One of the famous realizations of modern mathematics (called 'Godel's theorem') deals with the inevitability of such paradoxes.

The simplest example, which appears in many different guises, is the following. Determine the truth or falsehood of the statement: "This statement is false." Is it true or is it false? Let us suppose it is true; then by the very assertion *it must be false*. On the other hand, suppose it is false; then, we reach the contradictory conclusion that it must be true! There is no way by which one can 'logically' decide on the truth or falsehood of the above statement. Logic has tied itself up in knots.

The fact that such "undecidable" statements exist is of deep significance. There are many conjectures in mathematics which have remained unproven for ages. [Take, for example, Goldbach's conjecture, which states that 'every even number larger than two can be expressed as the sum of two primes'; eg. $4=3+1$, $12=7+5$, $36=17+19$ etc. Nobody has proved or disproved this conjecture.] A conventional mathematician working on a problem tacitly assumes that he can decide about the truth or falsehood of the conjecture by a "logical" process. The existence of 'undecidable' statements adds a new dimension to the issue; it is possible that a conjecture cannot be proved or disproved within the system of logic we are accustomed to!

Another powerful logical paradox is that of 'Surprise Examination'. [Like the previous one, this has also appeared in many different versions since the first discussion in *Mind*, July, 1951 by Michael Scriven.] A lecturer tells his class on a particular Friday: "Now, listen! I am going to give you a surprise test on one of the days next week. I will consider the test to be a 'surprise', if you have no way of logically deducing on the previous day that there will be an exam on that particular day. Go home and study, best of luck!" Over the weekend, it dawns on the smart alec, Johnny, that the lecturer simply cannot give such a test! "Look at it this way," reasons Johnny, "he cannot give the test on Friday for the following reason: If he hasn't given the test until Thursday, then, I know for sure that the test will be on Friday; it will no longer be a surprise. So Friday is out. But by the same reasoning, he can't give the test on Thursday! Suppose he hasn't given the test until Wednesday; then only Thursday and Friday remains. We have already seen that Friday is out, leaving only Thursday. So you see, if he hasn't given the test until Wednesday, it has to be on Thursday and it will no longer be a surprise. Well, by similar reasoning, it can be shown that the test can't be on Wednesday, Tuesday or Monday." Delighted with his logical prowess Johnny goes for a trek on the weekend.



In accordance with his "logical reasoning", there was no test on Monday. Then came the bombshell: the lecturer gave the test on Tuesday taking Johnny completely by surprise! It was a surprise test in the truest sense of the word! Where did Johnny's logic go wrong?

It is probably worth looking at the above question in a simplified manner suggested by Martin Gardner (*Further Mathematical Diversions*, Pelican books, 1969, p. 16). Your friend shows you the 13 cards in the suit of spades. He chooses one of these cards and puts it face down on the table. You are asked to name slowly the 13 spades starting with ace and ending at king. Each time you fail to name the card on the table, your friend will say 'no'; when you name the card correctly he will say 'yes'. "I will offer you thousand rupees against one", says your friend, "that you will not be able to deduce the name of the card before I say 'yes'." Well, what card do you think your friend would have put down? It can't be the king of spades; after naming upto queen of spades and getting 'no' as answers you can guess the card and win the bet! By similar "reasoning", it can't be queen, jack, 10....etc.! The logic is airtight; and still you don't have the foggiest notion what card is on the table!

I will let the reader wade through the logical labyrinth on his own and would be glad to discuss in this column any interesting new insight.

Fortunately, the method of logic works most of the time! The problems below can all be solved by straightforward reasoning. Send your answers and comments to SCIENCE TODAY. It is, of course, not necessary that you should solve all of them.

(1) One hundred politicians attended a convention. Each politician was either crooked or honest. You are given the following two facts: (i) At least, one of the politicians was honest, (ii) Given any two politicians, at least one of the two was crooked. Can you find how many of the hundred politicians were honest?

(2) A woman answers questions either always truthfully, or always falsely, or alternates truth and falsehood. How can you determine in two questions (each to be answered 'Yes' or 'No') whether she is a truther, liar, or alternater? (If you succeed in cracking this, see whether you can do the same with a single question).

(3) This is an old-timer which has appeared in print scores of times; but if you haven't seen it before, it is well worth trying:

Prof. Matics invites Prof. Logic, a colleague from abroad, for dinner; the following conversation ensues:

Prof. Matics: I understand you have three children; how old are they?

Prof. Logic: The product of their ages (taken correct to the nearest integral number of years) is 36.

Prof. Matics: Come on, now! Surely, that is not enough information.

Prof. Logic: While coming to your house, I noticed that the sum of my children's age is equal to your house number.

Prof. Matics: Mm...m.. that is interesting, but, no, there still isn't enough information.

Prof. Logic: Here is one last bit. My eldest

kid, who is at least a year older than either of the other two, goes for swimming lessons.

Prof. Matics: Ha, say that! Now I know the ages.

Well, can you find the ages?

(4) Among the assertions made in this problem there are three errors. What are they?

1) $(2+2) \times 0 = 0$

2) $4 \div \frac{1}{2} = 2$

3) If all crows are black and all black things are beautiful, then all crows are beautiful.

4) $3\frac{1}{5} \times 3\frac{1}{8} = 9\frac{1}{40}$

5) $(-1) \times 0 = 0$

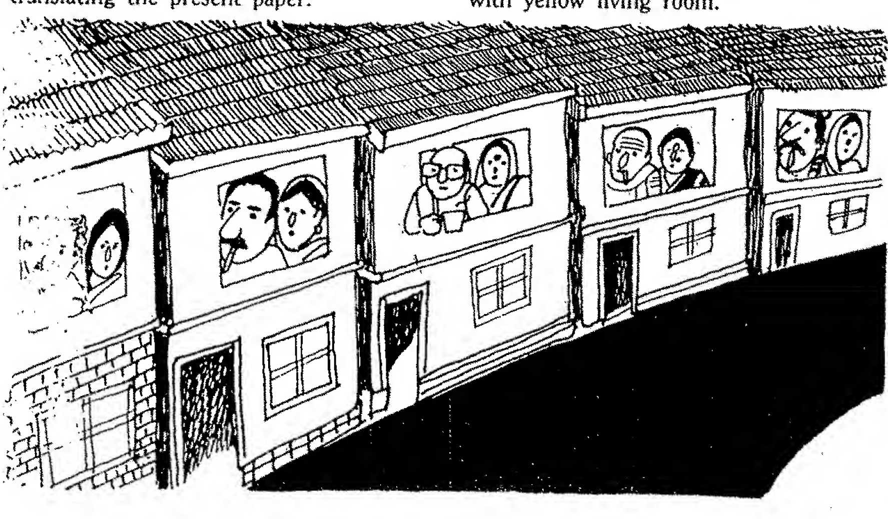
(5) Given the following set of ten statements, find out which of them can be true and which are false:

1. Exactly one statement in this list is false.
2. Exactly two statements in this list are false.
3. Exactly three statements in this list are false.
4. Exactly four statements in this list are false.
5. Exactly five statements in this list are false.
6. Exactly six statements in this list are false.
7. Exactly seven statements in this list are false.
8. Exactly eight statements in this list are false.
9. Exactly nine statements in this list are false.
10. Exactly ten statements in this list are false.

(6) Prof.Littlewood,the English mathematician who does not know French, has a paper published in a French journal. At the end of the paper, which is in French, appears the following notes (also in French):

1. "I am indebted to Prof. Risez for translating the present paper.
2. I am indebted to Prof. Risez for translating the preceding footnote.
3. I am indebted to Prof. Risez for translating the preceding footnote."

That's all! But shouldn't it go on and on? How could he stop after the third?



ILLUSTRATIONS BY: MUKUND TALWALKAR

(7) Five men from different states in India, live in five different houses. All the five houses are in a row on one side of the road. Each of their living rooms is painted in a different colour. Each man smokes a different brand of cigarette, and is married to a different(!) girl, and has a favourite drink of his own.

1. The Keralite lives in the house with the living room painted red.
2. The Punjabi's wife is named Kavita.
3. Coffee is drunk in the house with green coloured living room.
4. The Bengali drinks tea.
5. The house with the green living room is immediately to the right (your right) of the house with white living room.
6. Shanthi is the wife of the man who smokes Charminar.
7. Four square is smoked in the house with yellow living room.
8. Milk is drunk in the middle of the five houses.
9. The Bihari lives in the first house on the left extreme of the five houses.
10. The man who smokes Panama lives in the house next to the man whose wife is Usha.
11. Four Square is smoked by the man living in the house next to the house where Jaya stays (Jaya, of course, is staying with her husband).
12. The Chesterfield smoker drinks orange juice.
13. The Tamilian smokes Scissors.
14. The Bihari lives next door to the house with blue living room.

Now, tell me, who drinks coconut water? And whose wife is Maya?

(8) This last question, probably, is the trickiest (It was invented by David Gale of California in 1976):

An umpire chooses any pair of consecutive integers (it could be 1 and 2 or 2 and 3, or 81 and 82 etc). A piece of paper with one of the numbers written in it is stuck to the forehead of a man (call him John) and a similar piece of paper with the other number is stuck to the forehead of the second man (call him Babu). Both John and Babu are honest and extremely intelligent. Each can see the number on the other man's forehead, but not the one on him. They also know that the numbers are consecutive. The umpire asks one man, say, John, whether he can deduce the number on his forehead. If he says 'yes', the game ends there. If he says 'no', Babu is asked; if Babu says 'no', then the question comes back to John and so on.

What is the outcome of this procedure? Will John or Babu be able to guess his number at some stage?

T. Padmanabhan

Dr. Padmanabhan is with the Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

How to propose to your favourite girl, logically

THERE are some practical applications to the logical paradoxes. Suppose you are planning to propose to your favourite girl and are afraid of a probable 'no'. The following strategy may work. Ask her the following three questions:

1. "Will you answer this and the next two questions truthfully in yes/no?"

If she says 'no' here itself then the method doesn't work; but then, do you really want to propose to a girl who may not be truthful in life? On the other hand, if she says 'yes' here, ask the next two questions.

2. "Will you give the same answer to this and the next question?"
3. "Will you marry me?"

If she says 'yes' to question 2, then she has to give the same answer to question 3 and you get your 'yes'. On the other hand, if she says 'no' to question 2, then again she is forced to answer 'yes' to question 3. Best of luck!

PLAY THEMES

Solutions—and some problems

THE questions discussed in the November and December 1985 issues have prompted much response from readers. I will discuss the solutions in this issue; any letter I received after January second week will be discussed in a subsequent issue. So don't be disappointed if you miss your name here!

Let us begin with the November issue. The first problem was to arrange hearts and spades suits so that the sum of the values of the cards is a perfect square. Clearly, 9,10 and J have to be paired with 7, 6 and 5. Since 6 is used, 3 must pair with K; similarly, since 5 is used 4 must pair with Q. Rest of the combinations are unique, giving the solution 1-8, 2-2, 3-K, 4-Q, 5-J, 6-10, 7-9, 8-1, 9-7, 10-6, J-5, Q-4, K-3.

The second question was to determine the most probable position for the first black ace in a pack of cards. Contrary to intuition, its most probable position is the top place! (This can be clearly grasped by looking at a reduced pack of three cards, consisting of two black aces and, say, a two. After shuffling, there are three equally probable arrangements: 2AA, A2A, AA2. Obviously, the first ace being on top has a probability of (2/3). In the pack of 52 cards, the probability for the top card to be a black ace is (2/52) (remember, there are two black aces). In order for the first black ace to be the second card from the top, we have to satisfy two conditions: the first should *not* be a black ace (probability 50/52) and the second should be (probability 2/51). Thus the chance for the first ace to be the second card is 50/52 x 2/51 = (2/52) (50/51), which is lower by a factor (50/51), compared to the probability for the ace to be the top card. In general, the probability for the first ace to be at the n^{th} position from top is (2/52) (52-n)/(51). This problem is discussed by A./E. Lawrence in *Mathematical Gazette, 53,* December 1969, p. 347-354.)

The arrangement required in the third problem is 41312432; it can, of course, be reversed. The more challenging task involving 27 cards has the following solution: 1, 8, 1, 9, 1, 5, 2, 6, 7, 2, 8, 5, 2, 9, 6, 4, 7, 5, 3, 8, 4, 6, 3, 9, 7, 4, 3. (I know of two more solutions but there are no simple ways for construction). The solution with four pairs of cards is unique. This problem was posed by C. D. Longford in *Mathematical Gazette* (42, October 1958, p.228). Subsequent discussions can be found in the same journal in later issues (43, December 1959, p.250-55; 55, February 1971, p.73).

The fourth problem involves a somewhat intricate reasoning. Suppose ten cards of a suit (Ace, 2, 3...10) are arranged in some arbitrary order in a row. We shall attach to each card two numbers, X and Y, defined as follows: X denotes the maximum number of cards on the right side of a particular card (including that particular card) which are in ascending order; Y denotes the maximum number of cards on the left of the chosen card which are in descending order. It is easy to see that no two cards in the row can have both its X and Y numbers the same. Suppose we have found an arrangement of ten cards such that there are not more than three cards in ascending or descending orders. Then the X and Y for any of the cards can only be 1, 2 or 3. Altogether X and Y can only cover 3 x 3 = 9 arrangements. But we have seen that all the ten cards must have unique X and Y values! Thus it is impossible for (X, Y) to be less than 3. There should be at least one ascending or descending order with at least four cards. A good discussion of this problem and its generalisation can be found in *The Mathematical Sciences* by G.C. Rota (MIT Press, 1969).

The last problem, of course, is quite easy. Just deal the bottom card of the deck to yourself and proceed in the anticlockwise direction dealing from the bottom of the deck.

None of the readers who have sent their answers have solved the fourth problem. Mr. Aby Mathews of Trichur and Mr. M. N. Deshpande of Aurangabad have sent the most complete solutions to other problems. Mr. Altekar of Pune has sent correct solutions to the first and fifth problems. He has also sent a general method of solution to the royal magic square of cards discussed in the November issue. Congratulations!

The first problem in the December issue is quite easy if one goes about it in the correct way. First, balance a red and white against a black and white. If the scales balance, we know that there is a heavy ball and a light ball in each pan. Just remove the coloured balls, leaving two white balls, one on each side. The position of the pan will tell you which white ball is heavier and thus which of the coloured balls (among the ones you removed) is heavier, allowing complete identification in two weighings. If the scales do not balance, we know that the side that went down must have the heavier white ball. Now weigh the original red ball against the *mate* of the original black ball. The result of the weighing is sufficient to identify the balls completely. This solution, suggested by C. B. Chandler, is discussed in *Mathematical Circus* by Martin Gardner

Mrs. Parameswaran of Bangalore has alone attempted and solved this problem. There are many other ways of tackling these weighing problems. One approach and generalisation to more complicated cases was sent to me by Mr. N. M. Dongre of Bombay. If I get a chance, I will discuss weighing problems in detail in a later issue.

The correct answer to the second problem is 21 squares. (As shown in Fig.1 here, there are four squares of the type BEJF, four squares of type ADLG, four squares of type CELK and two of type AIMH. These are in addition to the nine small squares.) This problem has an interesting history. It originally appeared in a book *Puzzles Old and New* by Angilo Lewis (1893) and also in an article by the same author in a 1909 issue of the *Strand* magazine. He gave the answer as 17. H. E. Dudney, the noted puzzlist, "corrected" Lewis and gave the answer as 20 in his *Amusements in Mathematics* . Actually, there are 21 and Dudney gave this revised figure when he reprinted the book.

The following construction (see Fig. 2)

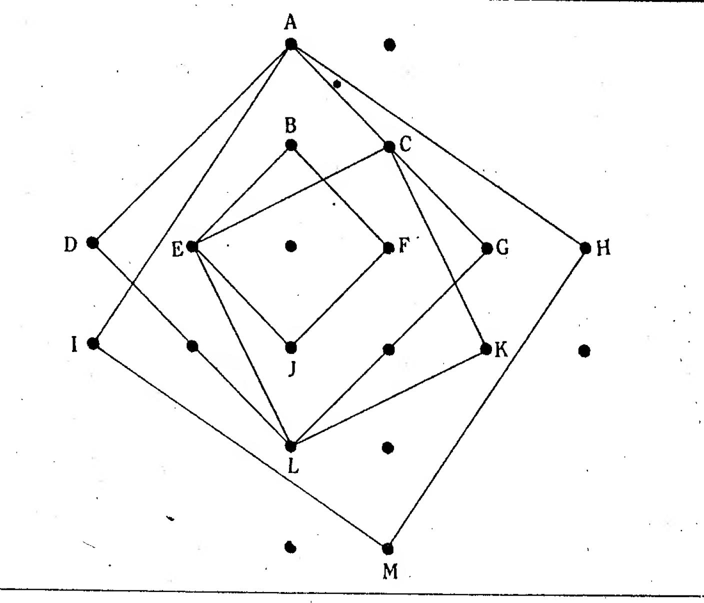


Fig.1

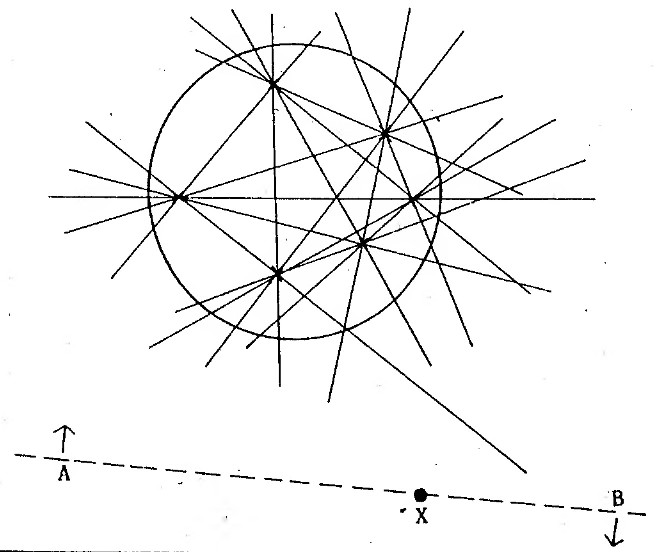


Fig.2

will show how to draw a straight line dividing a finite set of points into two equal parts. For simplicity, we consider just six points, even though the proof applies to any finite number (however large) of points. Draw every line determined by every pair of points. Pick a new point X which is outside the circle and does not lie in any of the lines. AXB is any line through A. Rotate this line about X as shown in the figure. Clearly, it will pass through one point at a time. It cannot pass through more than one point at a time because A does not lie in any of the straight lines containing the points. Just after the rotated line AXB has passed through half the original points, it will divide a set of points in half (for the original version of this problem, see H. Wills in the *Mathematical Gazette*, May 1964). An interesting alternative solution has been sent by Mr. R. D. Mathur of Pune.

The fourth problem requires knowledge about number bases other than 10. In the usual notation, 243 actually stands for the number, $3 \times 10 + 4 \times 10^1 + 2x\ 10^2$ (We talk of unit's place, ten's place, hundred's place, etc.). One could very well use bases other than 10 to represent numbers. The same number, 243, in base 6, for example, will denote $3x6+4x6^1+2x6^2 = 99$ (in base 10). The figures 10,11,12, etc given in question four represent the number 16 (base 10) in bases 16, 15... etc; with this reasoning (admittedly perverted), the missing numeral is 121. Note that 121 (base 3) = $1x3+2x3+1x3^2 = 16$ (base 10).

The unique answer to the fifth problem is 6210001000. For less than ten digits, the problem has the following answers: 1210,2020; 21200; 3211000; 42101000 and 521001000. If one is working in a base n (rather than base 10) (n >6), then one can write a general solution in the form x21(0......0)1000 where x=n-4 and there are (n-7) zeroes inside the brackets (these solutions by F. Rubin are discussed in *The Journal of Recreational Mathematics*, 11, 1978, p.76). A very complete analysis of this problem and its generalisation was sent by Mr. T. R. Mohan, TCS, Bombay.

The desk calendar in problem 6 appears to be an impossibility at first sight. There are four hidden faces in the left cube and three in the right. To get 11 and 22, we need 1 and 2 in both cubes. Again in order to produce all dates from 01 to 09, we need zeroes in both cubes. Thus the left cube has 2,1,0, leaving three faces and the right cube has 3,4,5,2,1,0, leaving no blanks. But we have four digits, 6,7,8,9, to put! It looks like an impossibility until one realises that 6 can be reversed to obtain 9. Thus the left cube has 0,1,2,6,7,8 and the right cube has 0,1,2,3,4,5, making the desk calendar. (According to Martin Gardner, who discussed this problem in one of his columns, this cube was once patented by J. S. Singleton of U.K.).

In the way it is stated, the last problem can have a solution if the barber is a woman! (According to Random House dictionary, 'barber' means 'a *person* whose occupation is to cut hair, shave beards, etc). If the barber is a man, of course, we run into the famous barber's paradox. Those readers who knew about the original paradox were the first to jump to the wrong conclusion!

Mr. Rajeev of Trivandrum and Mr. Pradhan of Madras have sent correct, complete solutions to all the problems. Mr. Aby Mathews of Trichur has got all but the third problem correct; his approach to the third problem was imaginative but the answer was incomplete. Mr. Mathur (Pune) has solved all but the fourth, while Mr. Dongre and Mr. T. R. Mohan (both from Bombay) have solved all but the last one. Mr. R. Mohan of Neyveli has got all except the third and fourth correct. Mr. Unni Krishnan of Kuwait and Mr. Anjan (Ravimutula) have solved four problems correctly. Mr. A. Choudhry, Mr. S. Dhanani (Bombay) and Mr. Kalyanaraman (Madras) have sent correct solutions to three problems each. Congratulations!

We conclude with some assorted problems for the reader to try:

1. A nine-digit number is formed by using the digits (1,2,...9) each appearing once and only once. This number has the following curious property. The first two digits (reading from the left) make a number divisible by 2; first three digits make a number divisible by 3;... and so until the entire number is divisible by 9. Find the number.

2. 'Persistence' of a number is defined to be the number of steps required to reduce it to a single digit by (i) multiplying all its digits to obtain a second number, *(ii) then* multiplying the digits of the second number to obtain a third number and so on. An example will make this clear. Consider the number 88; multiplying the digits we get 8x8=64; multiplying again, we get 6x4=24 and finally 2x4=8. Thus it took three steps to bring 88 to a single digit; we say that 88 has the persistence of 3. Question: Find the smallest numbers with persistence of 1,2,3,4 and 5.

3. Here are three problems which must be familiar to all brain-teaser buffs:

i) Two beakers contain equal amounts of water and wine. Some amount of water is transferred to the wine, the beaker is stirred well, and the same amount of mixture is transferred back. Is there now more water in wine or more wine in water?

ii) A cylindrical hole, six centimetres long, is drilled through a sphere. The material in the cylindrical portion and in the 'caps' is removed. What is the volume of the remaining portion?

iii) A race track is in the form of a ring (i.e., a region between two concentric circles). The longest straight line path, confirmed within the track has a length of 100 metres. What is the area of the race track?

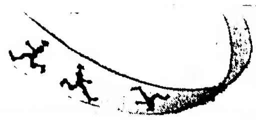
Now this question: What is common to the above three problems?

4. In a recent conference in Delhi, more than thousand delegates participated. As to be expected, there were many handshakes during the welcome dinner. Prove that the number of scientists who shook hands an odd number of times is even.

T. Padmanabhan

Dr. Padmanabhan is with the Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

Cut it out, will you?



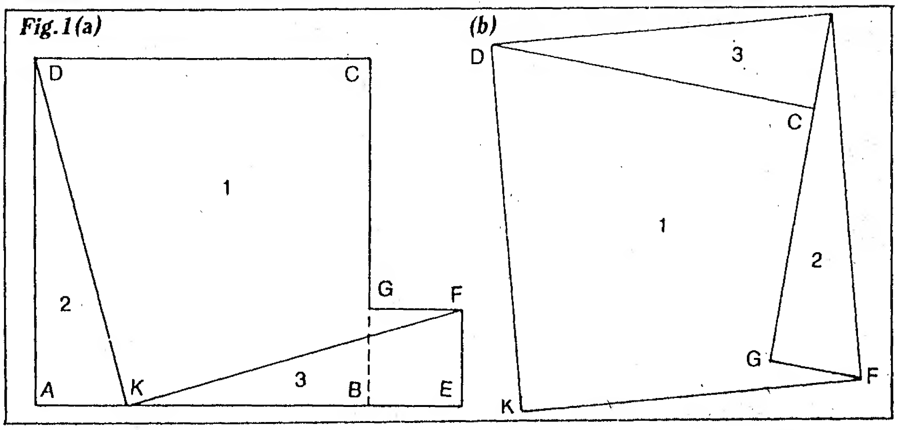
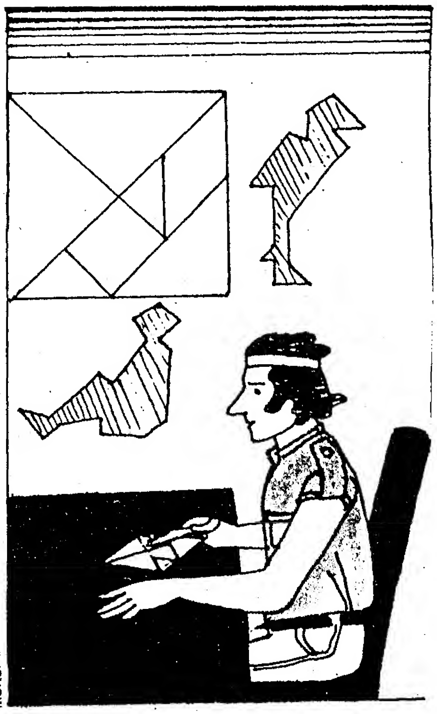


Fig. 1(a)

(b)



ake a look at Fig. 1(a). It consists of two squares ABCD and BEFG on a common base, forming a six-sided figure AEFGCD. Now choose a point K such that AK=EF and join KD, KF. Clearly, the triangles AKD and KEF are congruent. If you now cut along KF and KD and rearrange the pieces as in Fig. 1(b), you get a larger square with DK as the side! The sum of the areas of the squares in Fig. 1(a) is equal to the area of the square on the hypotenuse DK—a simple proof for the Pythagoras' theorem!

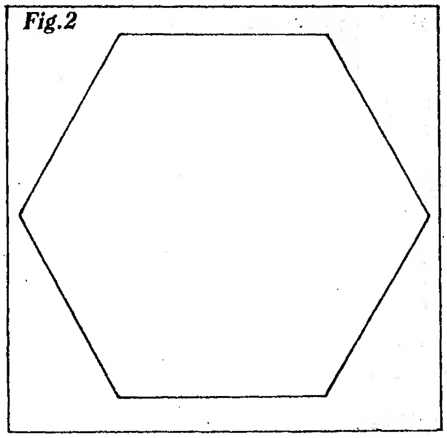


Fig. 2

Such geometrical dissections—which involve cutting out a given figure and rearranging it—have a peculiar charm of their own. The idea, of course, is that the original figure must be cut into as few pieces as possible and rearranged to form another figure. There is no general technique for tackling such problems. In most cases, it has been impossible to prove that a particular solution is indeed the one with the smallest number of pieces. There is always scope for improvement, and for alternative solutions.

Let us begin with some simple examples. The regular hexagon in Fig. 2 can be cut into five pieces which can be rearranged to form a square. The reader should probably try to crack this before turning to page 65 for solution; the solution given, however, is not unique! There is an entirely different way of dissecting the hexagon into five pieces which can again be rearranged to form a square. Can the reader find it before we discuss it in a future issue?

The level of imagination required to do the dissection problems has often led to some "almost correct" answers. One classic example is related to the geometrical shape shown in Fig. 3(a). This figure (called a "mitre") is formed by removing a triangular portion from a square of size, say, 84 cm. Sam Loyd, the famous American puzzlist, posed this problem: "Cut this figure into four pieces which can be rearranged to form a square". He proceeded to solve it in the following way. Cut the figure along the lines shown in Fig. 3(b). Rearrange pieces 1 and 2 (as shown in Fig. 3(c), forming a rectangle FBCE. Now move piece 4 one step lower to obtain the square in Fig. 3(d). Sam Loyd's solution is quite elegant except that it is wrong! Notice that the rectangle in 3(c) has the dimensions 84 cm × 63 cm. The steps are easily seen to have a height of 10.5 cm and a width of 12 cm. So the final fig. in 3(d) has sides (84-12) = 72 cm and (63+10.5) = 73.5 cm; this is not exactly a square!

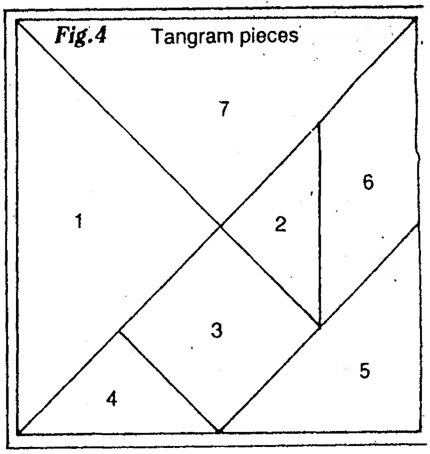


Fig. 4 Tangram pieces

This flaw was pointed out by H. E. Dudney, the British puzzlist. He produced a

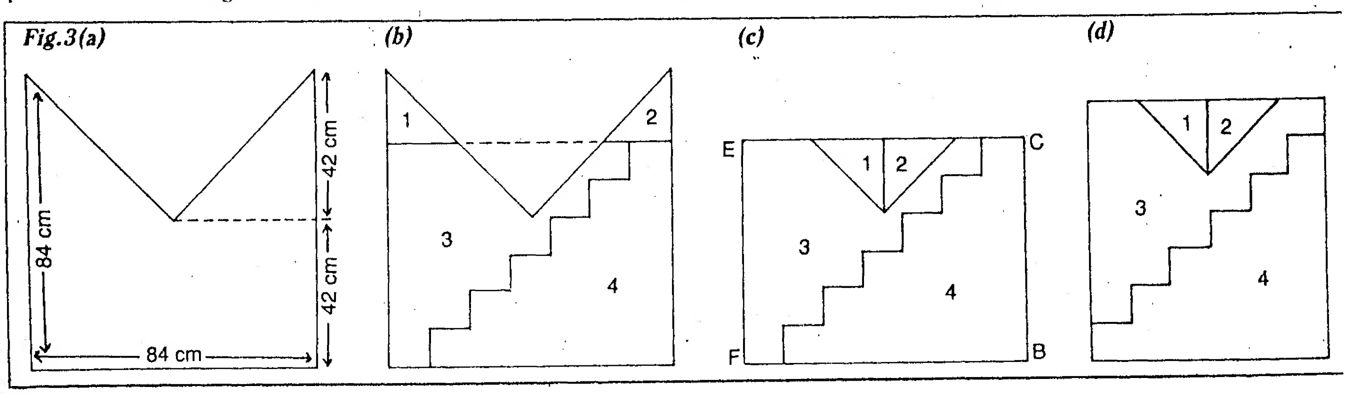


Fig. 3(a)

(b)

(c)

(d)

dissection of the figure in 3(a) into *five* pieces that can really be arranged to form a square. Can the reader repeat Dudney's feat? (See page 65 later). Incidentally, nobody has produced a four-piece solution to this problem yet. Harry Lindgren—who is an authority on dissection problems—has achieved the next best: Given *two* identical mitres, each can be cut into four pieces each and the pieces can be rearranged to form *two* congruent squares. The reader is invited to attempt this challenging task.

There are many games and puzzles involving dissections. Probably the most elegant one is what is called 'tangrams'. Tangram is a recreation (or 'art' or 'puzzle' or 'game') consisting of producing all kinds of figures and shapes from seven pieces, originally cut from a square (see Fig. 4). Cutting the figure along the lines shown in Fig. 4 produces seven pieces—five triangles, one square and one oblong. An astonishing variety of shapes can be produced by the arrangement of these pieces. The examples in Fig. 5 should convince the reader of the extraordinary scope and liveliness that these seven pieces offer.

There are many puzzles related to tangrams, which are essentially dissection problems. One of them (devised by Dudney)

Fig.5

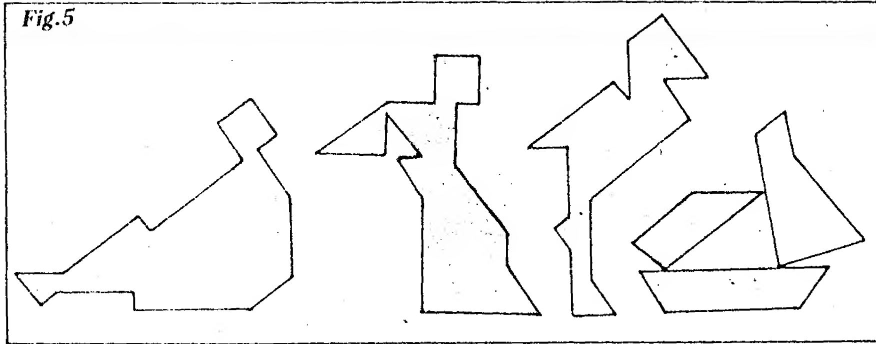


Fig.6(a) *(b)*

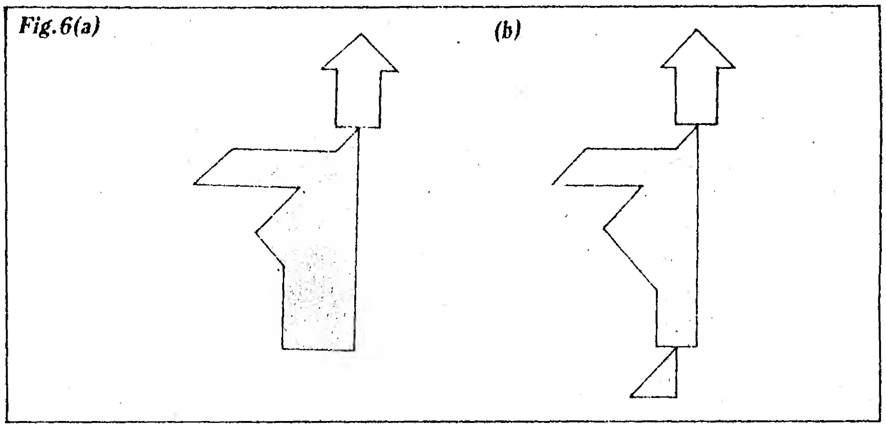
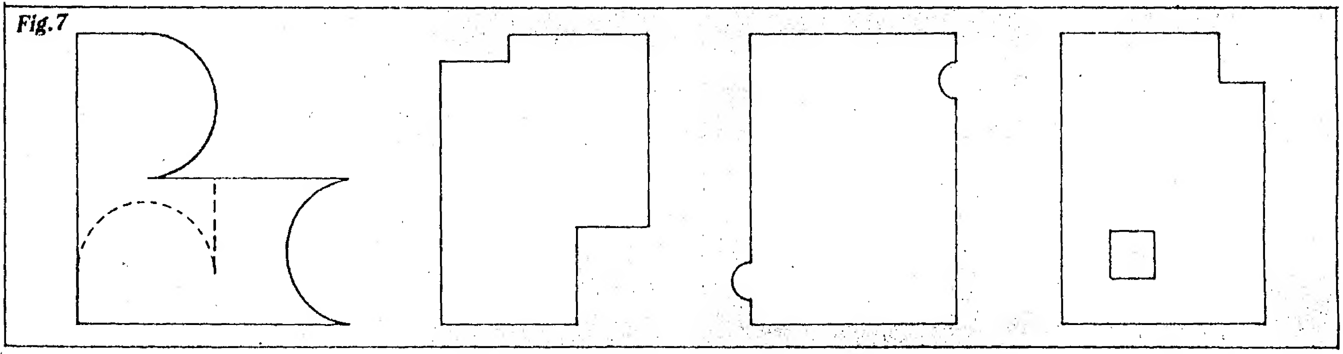


Fig.7



is the 'case of the missing foot'. See Figs. 6(a) and 6(b). Both are made by using all the traditional seven pieces of tangrams, and are supposed to represent a 'gentleman'. But how come that the gentleman has his foot in one but not in another! I would suggest that the reader make a set of tangram pieces before attempting this problem.

Here are some more problems for the reader to cut out:

1. A special class of dissection problems which can be annoyingly difficult are the 'bisection problems'. Some particular shape is given and the aim is to cut it into two congruent pieces with a single cut. Fig. 7 shows a series of shapes which the reader can try his hand (knife!) at. As a starter, 7(a) is solved for you: the single cut along the dotted line will divide the shape into two congruent parts.

2. How many pieces can a circle be cut into by five straight cuts? (If successful, try to answer the question for n straight cuts. Trick solutions, like keeping pieces on top of one another before cutting again, etc, are not allowed.)

3. This last question, probably, is the simplest. Rearrange the five pieces in Fig. 8—four in one large square and the fifth being the separate suare—so as to form one single big square.

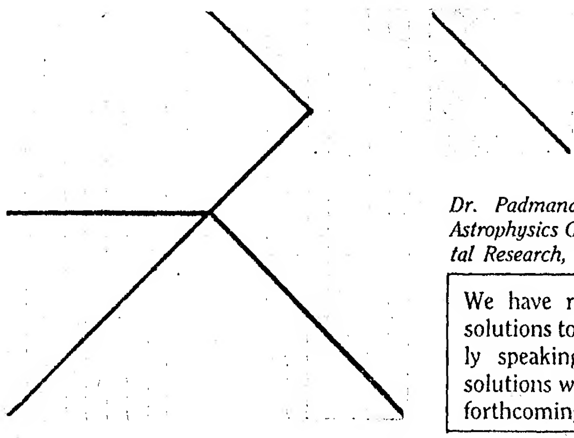


Fig.8

T. Padmanabhan

Dr. Padmanabhan is with the Theoretical Astrophysics Group, Tata Institue of Fundamental Research, Bombay.

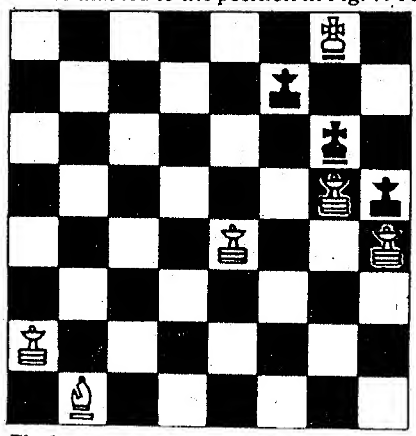
We have received a large number of solutions to the 'Playtheme' on "Logically speaking..." (February 1985). The solutions will be discussed in one of the forthcoming issues.—*Ed.*

Answering again—chess and logical problems

THE chess and logical problems (January and February) brought a large number of responses. Here, I will comment on all the solutions received by the middle of March. Solutions received later as well as some late solutions to earlier problems will be discussed in a subsequent issue. (Please do not send self-addressed envelopes because it would be difficult to communicate *individually.)*

In the January chess problem, there was a mistake in Fig. 6. A black pawn at black's Q4 was inadvertently omitted. The answer given on page 75 of that issue makes sense only with this pawn. I thank the readers who took the trouble to point this out.

The first question was to retrace the moves that led to the position in Fig. 7. To



BAIJU PARTHAN

Fig.1

arrive at this position, start with the position in Fig. 1(of this issue!) which could have been reached easily in a game. Now the following moves will produce Fig. 7 given in the January issue:

1. P-K5 ch P-B4
2. PxP ep. ch. mate

The crucial feature, of course, is the 'en passant'. capture.

The second problem was to identify the position of the white king in Fig. 8. The white king must be at QB-3. To arrive at this position, start with the pieces set as in Fig. 2 and proceed with these moves:

1. B-Q4 ch
2. P-B4 PxP ep db. ch.
3. KxP ch.

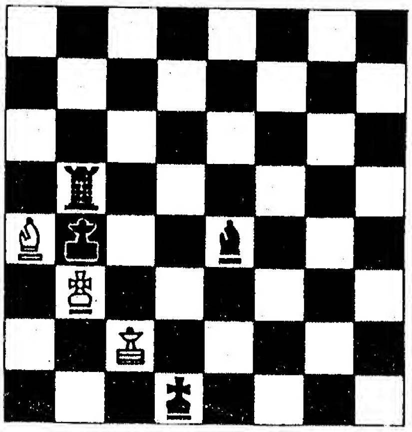
Once again 'en passant' capture plays a crucial role. This problem is one of Raymond Smullyan's remarkable creations. (More such problems can be found in his book, *The Chess Mysteries of the Arabian Knights,* Alfred Knoff, New York, 1981). It is easy to see that the solution is unique.

In problem 3, the only move that will not result in immediate checkmate is 1. R-QB6 ch, allowing the reply 1... RxB. Some readers (for example, R.A. Deshpande, K. Mahesh, Y.D. Altekar) claimed that the position in Fig. 9 was illegal because it shows two white bishops kept in white squares. This claim is wrong. It is perfectly feasible to arrive at a chess position in which two bishops are *lodged in the same coloured squares. It* merely means that a pawn reaching the eighth rank has been promoted into a bishop sometime during the game! It may be a stupid move, but it is perfectly legal.

There were three more problems in the same issue presented in the main text itself. The first one was to cover a chess board from which two squares (one black, one white) were removed by a set of 31 rectangular pieces. This can always be done. To see this, consider Fig. 3 here. The heavy lines divide the chess board into a closed path along which the cells lie like black and white beads in a necklace. The removal of black and white squares from any two places will cut the path into two open-ended segments (or one segment if the squares are adjacent in a path). Since each segment must consist of an even number of squares, each segment can be covered completely by the (2×1) pieces!

The second problem was to show that re-entrant knight tours are impossible on a 4×n board. The proof is illustrated with a 4×7 board (see Fig. 4). Any such board can be labelled by four colours A, B, C, and D as shown. Note that in a closed tour there will be one C before and after every A. There are equal numbers of A and C cells all of which must lie in the path. But this can be achieved only if B and D cells are completely avoided. (Note that once a

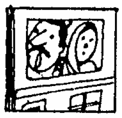
Fig.2



leap from C to D is made, there is no way of getting back to A without landing on another C). This solution is originally due to S.W. Golomb.

The last question was to keep eight queens in the chess board such that: (I) they do not attack each other, and (b) they are not in a straight line. The solution is shown in Fig. 5 here.

Correct answers to all the three end problems were sent by Janakiraman of Madras (I hope I got the name right; the signature was a bit difficult to unravel) and Awani Kumar of Delhi. Awani Kumar has also answered the eight queens prob*lem correctly. Congratulations! The third* problem was solved by A.R. Deshpande (Banglaore) and S. K. Shyamsukha (Hooghly) as well. Many readers have concluded that the other problems cannot be solved. I hope the solutions have convinced them; further discussions, comments, etc are welcome.



Logical problems

Let us now go to the logical problems in the February issue:

(1) The answer is: one honest politician

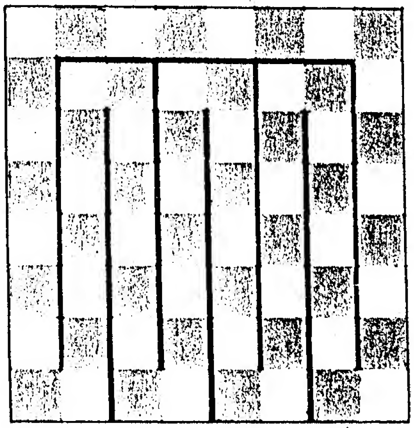


Fig.3

and 99 crooked ones. (Suppose there were more than one honest politician, then clearly statement (2) will be violated.)

(2) The simplest method is to ask twice any question with a definite 'yes' answer (like, say, "Do you have two eyes?"). The truther will say 'yes-yes', the liar will say 'no-no' and the alternator will say 'yes-no' or 'no-yes'.

Can we do it with a single question? Since the answer to a single question can be only 'yes' or 'no', it is impossible to decide between three possibilities *honestly.* But one can think of some trick questions which will put one of the three types into endless confusion. Probably the readers would like to have a second try at it!

Bihari	Bengali	Keralite	Punjabi	Tamilian
Usha	Jaya	Shanthi	Kavita	Maya
Yellow	Blue	Red	White	Green
Four Square	Panama	Charminar	Chesterfield	Scissors
Coconut water	Tea	Milk	Orange	Coffee

(3) The correct answer is (9,2,2). The reasoning is as follows: Since the product of the ages is 36, the ages must be one of the following: (a) (1,2,18), (b) (1,3,12), (c) (1, 4, 9), (d) (1, 6, 6), (e) (2, 2, 9), (f) (2, 3, 6), and (g) (3, 3, 4). In each of these cases, the sum of the ages are: (a) 21, (b) 16, (c) 14, (d) 13, (e) 13, (f) 11, and (g) 10. Note that this sum is different in all cases except (d) and (e). Now Prof. Logic has told Prof. Matics that the sum is equal to the house number of Prof. Matics. If Prof. Matic's house number was 21, 16, 14, 11 or 10, he would have immediately identified the ages (for example, if the house number was 11, then it must be case (f) with (2, 3, 6) as ages; no other set of numbers have the product 36 and the sum 11). Since Prof. Matics could not determine the ages at this stage, his house number must be 13 and the agest must be either (1, 6, 6) or (2, 2, 9). The last statement ("the eldest is at least a year older than the others") rules out (1, 6, 6), leaving (2, 2, 9) as the correct answer.

(4) Statements 2 and 4 are definitely wrong. Since there are only two errors in the statements 1-5, the assertion that there are three errors is also in error. This is the 'conventional' answer to this question (Do you agree with it?).

Many readers argued that crows are not "things" and hence the third statement is wrong: This is a good alternative!

(5) Only the ninth statement is true. Clearly, it is not possible for more than one statement in the list to be true. Suppose the n^{th} statement is true. Then (10-n) statements must be true. Setting $10-n=1$ gives $n=9$.

(6) Comments 1 and 2 were rendered into French by Prof. Risez. Comment 3 was copied by Prof. Littlewood himself from comment 2. Obviously, there is no need for any more thanks!

(7) The correct identification of the favourite drink, cigarette, etc, of the five men is given in the table above. (The reasoning is tedious but straightforward.) As the table shows, the Bihari drinks coconut water and the Tamilian is married to Maya.

(8) Let H be the guy with the higher number and L be the guy with the lower number. H always identifies the numbers first. If the numbers are n and $n-1$, and H is asked first, then H can guess on the $(n-1)^{th}$ trial if n is even and on the n^{th} trial if n is odd. If he is asked second, he wins on n^{th} question if n is even and $(n-1)^{th}$ if n is odd.

Fig.4

B	A	B	A	B	A	B
D	C	D	C	D	C	D
C	D	C	D	C	D	C
A	B	A	B	A	B	A

Let us see the argument behind the above conclusion. Suppose the numbers are 1, 2. If H is asked first, he will say 'yes'. If L is asked first he will say 'no'. Then H, seeing the number 1, will say 'yes' to the second question.

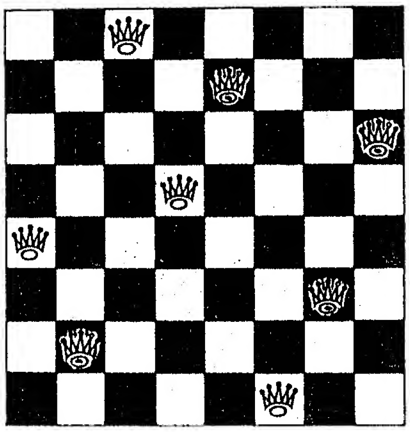


Fig.5

Suppose the numbers are 2, 3. If H is asked first, he says 'no'; L also answers 'no'. This answer proves that L does not see 1 on H, letting H know that he is 3. Clearly, he answers 'yes' to the third question. Similarly, if L is asked first, H can guess on the second question.

Suppose the numbers are 3, 4. If H is asked first, he says 'no', L also says 'no'. H can now reason: "If I am 2, the game is reduced to preceding case 2, 3 with L asked first. Therefore, L would say 'yes' to the second question. Since L said 'no', it proves that I am 4". A similar reasoning applies to the case of L being asked first.

Continuing in this way, one can always reduce the situation to a previously solved case. This allows one to arrive at the answer given above.

Among the large number of readers who sent the answers, N. M. Dongre (Bombay), V. Singhal (Lucknow) and R. V. Pradhan (Madras) have got all the answers correct. Close runners-up were: P. Mohommad (New Delhi), Nandeesh (Ketrinagar), P. S. Narayan (Calcutta), H. L. Patel (Ahmedabad) and Shyamsukha (Hooghly). The following readers got 5 to 6 answers correct: S. Babbu (Cuddalore), P. Bhattacharya (Calcutta), Sreerupa Das (Calcutta), Muraleedharan (Salem), Ms. T. Bhagwat (Jabalpur), D. Jana (Calcutta), Mrs. Malathy Sethuram (Madras), Mrs. K. Vanishree (Machilipatnam) and S. Rao (Mysore). Congratulations! The following readers have sent correct answers for 3-4 problems. Rajarishi Sinha (New Delhi), V. Madhok (Faridabad), K. Anjan (Calicut), A. Joshi (Nagpur), N. Swaminathan (Salem), R. H. Mhatre (Vapi, Gujarat), Kiran Mehta (New Delhi), Ms. S. N. David (Bombay), and George Kutty (Ellapally).

Now, here are a few assorted problems for the reader to attempt:

1. A chessboard has 1 cm squares. What is the radius of the biggest circle that can be drawn on this board in such a way that the circle's circumference is entirely on black squares?
2. Find four different digits, excluding 0, which *cannot* be arranged to make a four-digit number divisible by 7.
3. Which common word in English language is invariably pronounced wrong by the Doordarshan crew?

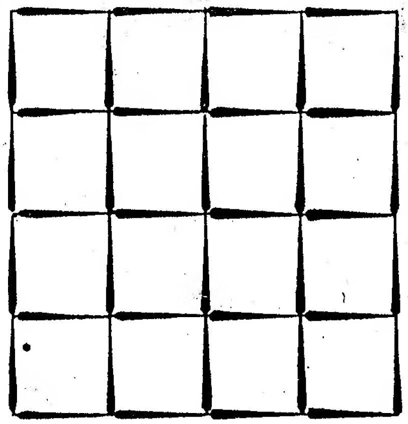


Fig.6

4. Figure 6 here shows a pattern made of toothpicks. Question: Remove the minimum necessary number of toothpicks, breaking the perimeter of all the squares in the figure. (All the squares, of course, means 16 small, 1-unit squares, nine 2-unit squares, four 3-unit squares and a large 4-unit square.)

T. Padmanabhan

Dr. Padmanabhan is with the Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Bombay.

PLAY THEMES

IT'S ONLY WORDS

T. PADMANABHAN

WE have confined ourselves to topics in recreational mathematics in this column so far. This time, we are stepping out slightly and exploring teasers based on the English language. The approach and ingenuity required to tackle and solve these is of a logical nature.

Many of the curiosities arise when numbers are spelt out in plain English. For example, have you noticed the following feature? Every positive integer—when spelt out—has one letter in common with the next integer. 'One' and 'Two' have letter 'o' in common, 'Two' and 'Three' have letter T, 'Three' and 'Four' have letter R and so on *ad infinitum*!

BAIJU PARTHAN

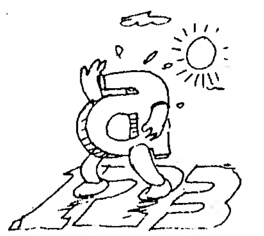
A more sophisticated example is the following: write down, in plain English, any sentence that springs to your mind: say, ' Playthemes is a lousy feature '. Now write down—in English—the number of letters in your sentence. Here it is "Twenty five". Next write down the number of letters in this sentence, and so on. You will get, ' Ten ', ' Three ', ' Five ', ' Four ', ' Four '... The process iterates very quickly to ' Four ' which reproduces itself. As the reader would notice, it really does not matter what the initial sentence was! Why?

Here is an assorted pack of questions—some easy, some difficult—based on words and numbers.

★

If you are asked to write down one hundred words without the letter 'a' appearing in it, how long will it take for you? (Try it before reading further!)

Actually, it is very easy. Just write down 'zero', 'one', 'two',... upto ninety-nine and you're through (well, probably not quite; but you can use hyphens in twenty-one etc., and try to get away!). It's really surprising that the letter 'a' does not appear for a long time when the numbers are spelt out. That brings us to the question: what is the smallest integer which has 'a' appearing in it? Similarly what is the smallest integer in which 'b' appears? What about 'c', 'd'?



While we are on this topic, can you spot the letter which is absent in 0,1,2,3,... upto 99, but is present in all numbers from 100 to 999999?

★

Once we spell out the numbers, the 'natural' thing to do is to order them alphabetically. Consider, for example, the numbers 0 to 1000, spelt out and arranged alphabetically. The list will begin with eight, eighteen... etc. (We follow the convention of omitting 'and' in usages like eight hundred and eight; it simplifies matters.) Clearly, the last entry will be zero.

What is the next to last, that is, the one just above zero entry?

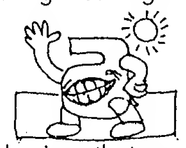
★

If we follow the conventions of the above problem and spell out *all* integers (not just 0 to 1000), which is the smallest number that will contain all the vowels plus 'y'?

★

One of the standard literary quiz questions is 'what is the longest word in the dictionary?'. Pretty weird creatures have cropped up as answers.

Let us assume that hyphenated words are allowed. Prove that there is no such thing as longest word. (In other words, show that you can always construct a legitimate word containing more letters than any given word.)



★

Find the next number in the sequence: 10^3, 10^9, 10^{27}, 10^2, 10^0... (Hint: this is a pretty tough one; but having solved the previous problem might help.)

One of the nasty trick questions goes as follows: "Rearrange these letters—OOUSWTDENERJ—to spell just one word. Not a proper name or anything unusual." The answer is fairly irritating.

Now for our question: "Rearrange the letters in NEW DOOR to make one word." One answer is obvious from the previous paragraph. But there is another—more original—solution; find that!

★

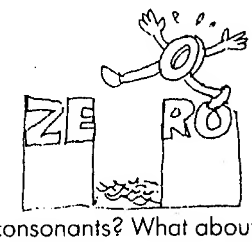
What is common to the following English words: deft, sighing, calmness, canopy, first, and stun?

★

Take the first thirteen letters of the English alphabet. They can be separated into four distinct classes as follows: class 1: A,M class 2: B,C,D,E,K class 3: F,G,J,L class 4: H,I. Determine the rule behind this classification and classify the rest of the alphabet.

★

(a) Find an English word in which five consecutive letters are vowels. (b) Find a word in which five consecutive letters are consonants? What about one with six consecutive letters being consonants? (c) Find a five letter word with only one consonant. (d) The word 'abstemiously' has a,e,i,o,u,y appearing in that order. There is at least one more word with this feature; find it. (e) Find a word which has the same vowel repeated six times. (f) What is the longest word you know in which no letter is repeated? (g) The word balloon has *doubled letters appearing* twice consecutively. Can you think of a word in which doubled letters appear thrice in succession?



★

Lastly, what is the cryptic message contained in the following code

A,B, ...

(Hint: this is not quite an honest question!)

Send your answers to SCIENCE TODAY. It is not—of course—necessary that you solve all of them.

MATH COMMUNICATION

T. PADMANABHAN

THERE is a special contest with 18 recreational questions in mathematics. Each question carries certain 'weightage' points, indicated at the end of the question.

And you have ATTRACTIVE prizes to win: Rs.500 to the top scorer, a three year subscription to SCIENCE TODAY to the second highest scorer and a one year subscription to the third highest scorer.

The points for all 18 problems total 140. You can, however, earn bonus points for any imaginative/unconventional solution, generalisations of the problem, or for any other illuminating discussion. Remember that it is the total score that decides the winner.

Send all the answers to SCIENCE TODAY so as to reach us before December 31, 1986. We will announce the results in April 1987 and discuss the solutions in some of the coming instalments of Playthemes.

Happy solving!

Teacher used to say readin' and writin' about 'rithmetic was the best way for rememberin' it. (Specially if they paid you for it too!)

COME RAIN OR SHINE

The Island of Weatherino does have fine weather. It never rains for more than one day in succession. Meteorological records over a very long period of time support the belief that a sunny day is certain to follow a rainy day. It was also noticed that after any sunny day the probability is always 1/2 that the following day will be rainy.

What number of sunny days per year occur in Weatherino most frequently? (5)

WEIGHT A BIT

"I have a curious bunch of 13 stones which may or may not weigh the same."

"What's so curious about them?"

"Well, to begin with, the weight of each stone can be expressed as an integral number of kg; i.e., there aren't stones which weigh in fractions like 2 1/2 kg etc."

"That doesn't sound so curious."

"Wait. It is also known that any 12 of them may be placed on a common balance with 6 on each pan in such a way that the balance will be in equilibrium."

"Oh I see. That's really interesting. In fact, I see that all your stones must be of equal weight."

Do *you* see why? (8)

SIMILARLY SYMMETRIC

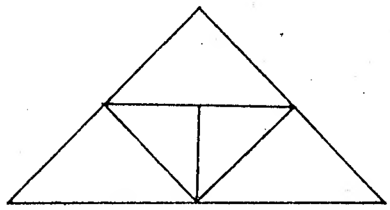


Fig. 1

Fig. 1 shows how to divide a triangle with angles 90°, 45°, 45° into 5 smaller triangles each similar to the original one in such a way that the resulting figure is symmetrical. Can you find another triangle that can be divided into 5 smaller triangles each similar to the original one so that the resulting figure is again symmetrical? (8)

WATCH OUT FOR ANGLES

In a normal clock, with an hour hand and a minute hand, the hour hand moves through 30° as the minute hand finishes one circle.

(a) How many times in a 12 hour period are the hands of a clock interchangeable, i.e., interchanging the position of the hands will again yield a possible clock reading? (An example will make this clear: at 2 O'clock, the hour hand will be at 2 and the minute hand at 12. Interchanging the hands will yield a position with hour hand at 12 and minute hand at 2. This position

PLAYTHEMES

PERSISTENCE AND OTHER QUALITIES

T. PADMANABHAN

THE response to problems posed in the March '86 issue of SCIENCE TODAY was overwhelming. Here we discuss answers to them.

The first problem was to find a nine-digit number made of (1, 2,....9) such that the figure made from the first n digits of this number is divisible by n. The unique answer is 381654729. The uniqueness can be easily proved by using the standard divisibility rules and some trial and error. This problem was solved correctly by 27 readers. Of them, complete analyses were sent by Sthanu Subramanium (Madras), M.S. Modak (Thane), Savitha Kumari (Baroda), N. D. Bhat (Madras), N. Kalyanaraman (Madras) and N.M. Dongre (Bombay). One persistent wrong answer contributed by nearly half a dozen readers is 921654387. (This is wrong because 92165438 is not divisible by 8; remember that a number is divisible by 8 only if the last 3 digits are divisible by 8.)

The second problem was to find the smallest numbers with the 'persistence' of 1, 2, 3, 4 and 5. These are respectively 10, 25, 39, 77 and 679. The chart alongside gives the smallest persistent numbers upto order 11. [Taken from N.J.A. Sloane, *Journal of Recreational Mathematics, 6,* 97 (1973.)] Using a computer program Sloane determined that no number less than 10^{50} (i.e., one followed by fifty zeros!) has a persistence larger than 11. In other words, all numbers less than 10^{50} can be reduced to a single digit by less than 11 successive multiplications of digits. Incidentally, this and the above problem offer interesting generalisations to bases other than 10. Examine them if you are interested.

Sixteen readers have sent correct answers to this problem. Many others got the smallest numbers with persistence 1, 2, 3 and 4 but failed to get the fifth. Full credits are due to N.D. Bhat (Madras), M. Kalyanaraman (Madras), and for N.M. Dongre (Bombay) for providing methodical analysis.

Many readers missed the essential point of the third question. The idea was not so much as to solve the three problems (i), (ii), (iii), but to find out what was common to the three problems! The three problems share the following common feature: they can be solved trivially if we assume that a unique solution exists. In (i) I have not specified how much liquid was transferred. If an answer can be found without this information then it should not matter what this amount is! Take the limiting case of all the water being mixed with wine. The amount of water in wine is same as the amount of wine in water in this case. If a unique solution exists then this must be the solution. Similar tricks work with (ii) and (iii). In (ii) assume that the sphere has

Persistence	Number
1	10
2	25
3	39
4	77
5	679
6	6,788
7	68,889
8	2,677,889
9	26,889,999
10	3,778,889,999
11	277,777,788,888,899

the radius of 3 cm. No volume will be lost when a 6 cm hole is drilled through such a sphere and hence the volume of the remaining material is $(4\pi/3) \times (3)^3$ cc or, 36π cub. cm. In (iii), we may take the radius of the inner ring to be zero making the track area to be $\pi(50 \text{ metres})^2$ or 2500π square m. Virtually every reader has bothered to solve the 3 problems, but most of them failed to state categorically what is common to the three brain teasers. The clearest statement and answer were from S.P. Pathak (Pune) and A. Kaushik (Jalandar). (I believe many other readers could see the similarity but they have failed to express it in unambiguous language.) Two readers, Rajesh Gopakumar (Calcutta) and J. Nandi (Calcutta) pointed out that the wine and water problem can be answered differently if 'more' is taken to mean more mass rather than more volume. M. Amarnath Murthy of Bhopal came up with the following common feature to the three brain teasers. "They have, all appeared in SCIENCE TODAY before." Good lateral thinking!

An interesting offshoot of the wine-water problem which is not so well known is the following: suppose that two vessels originally contained 10 litres each of wine and water. By transferring 3 litres of liquid each time and stirring well after each transfer, is it possible to reach a point at which the percentage of wine in each mixture is the same? It is probably fair to warn the reader that the question has a hidden subtlety in it.

The last problem—that of handshakes—can be easily done by the method of induction. The first handshake produces two 'odd persons'. From then onwards there are three types of handshakes: between two even persons, two odd persons or between odd and even. In each even-even shake, number of odd persons increases by two (since two chaps who had shaken hands even number of times before has now shaken hands odd number of times). Each odd-odd shake decreases the number of odd persons by two. Odd-even shake leaves the number of odd persons unchanged. Therefore, there is no way for the even number of odd persons to shift into odd number of odd persons. Thus at any stage, there will be an even number of persons who have shaken hands an odd number of times. This problem was done correctly by 10 readers, of whom, K. Srinivas (Bombay), Sthanu Subramanium (Madras) and Savitha Kumari (Baroda) came up with interesting alternative solutions. Among those readers who attempted all the problems, the best—i.e., almost all correct—answers were from K. Srinivas (Bombay), N.D. Bhat (Madras), V.K. Singhal (Lucknow), Rajesh Gopakumar (Bhopal) and M.S. Modak (Thane). Congratulations! ST

PLAYTHEMES

MAGIC FIGURES

T. PADMANABHAN

A CONVENTIONAL 'magic square' is an n × n square in which the integers 1, 2, ...n^2 are arranged in such a way that the sum of the numbers along the rows, columns and main diagonals add up to the same total. Forming such squares is an old amusement and these squares are often associated with mystical and religious ideas. The rule for drawing up such squares is easy to learn; once it is done, they become fairly boring, unless some additional features are introduced. I will discuss this time some puzzles and teasers which are—hopefully—unconventional enough to be interesting.

As examples of magic squares possessing unorthodox features, consider those in Fig. 1 and 2. Fig. 1

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig.1

8	1	6
3	5	7
4	9	2

Fig.2

shows a (4 × 4) magic square found in an eleventh century inscription in Khajuraho (also called 'pandiagonal' or 'Nasik' square). In addition to being 'magic' in the conventional way (viz., rows, columns and diagonals add to the same total), the square is also magic along the 'broken diagonals'. For example, the cells with (2, 12) and (15, 5) form 'broken diagonals'; so does the cells (2, 3, 15) and 14. As the reader can verify, the sum along these broken diagonals also lead to the same total.

We shall examine more variations in magic squares through a series of problems:

1. It is possible to obtain interesting variations in much simpler magic squares as well. Consider, for example, the (3 × 3) magic square (Fig.2) known to everybody. There are many different ways of arranging the digits so as to obtain a total of 15. To make the problem interesting, let us fix the position of 8 in the top-middle-cell (Fig.3). Question: can the reader put numbers in the remaining cells and construct a magic square with a total of 15? To begin with, can it be done by using the same digits as in Fig.2? Can it be done with any other set of integers with the condition that numbers in each cell should be different? Lastly, can it be done if fractions are allowed, still keeping the restriction that the numbers must be different?

Fig.3

2. Conventional magic squares, of course, are based on addition. Generalising this to other operations often leads to interesting diversions. For example, Fig. 4 shows a magic square based on subtraction. You get a constant value of 5 by subtracting the first number in a line from the second and subtracting the result from the third.

2	1	4
3	5	7
6	9	8

Fig.4

(For example, in the top line, subtracting 2 from 1 we get (−1) and subtracting this from 4, we get [4−(−1) = 5]. More simply, we can state the above rule as follows: add the first and the third number in any line and subtract the middle number; the result will be 5.

Can you generalise this further and obtain the 3 × 3 magic squares for multiplication and division? In a 'multiplication square', the product of the three numbers along each row, column and diagonal must be a constant. The 'division square' is made (analogous to the subtraction square) with the following rule: the product of the first and third numbers in any line divided by the middle number is a constant. (It is assumed that only positive integers are used).

3. Look at the seven strips of cardboard (Fig.5) each carrying the numbers 1 to 7. The problem is to cut these strips into the fewest numbers of pieces such that the pieces may be rearranged to form a magic square. (No tricks like turning figures upside down, cutting through a figure, etc., are allowed!). Of course, if you cut each strip into seven pieces, each piece carrying a single number, then it is trivial to form a magic square; but 49 pieces is far from the *minimum* that is possible.

1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7

Fig.5

4. So, far, we have been dealing with a square because the square defines the direction along which numbers are added—horizontal, vertical and diagonal. It is possible to think of other geometrical figures in which 'magic additions' can be performed. Take, for example, the hexagram of Fig.6, called

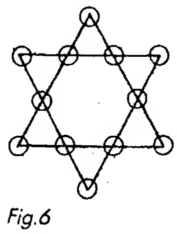


Fig.6

'Solomon's seal' or 'Star of David' and used in western occultism as well as oriental tantric worship. One may ask: is it possible to arrange the digits, 1, 2,... 12 in the circles such that the sum of digits along any of the lines of hexagram is a constant? It *is* possible, and the reader is invited to find it.

PLAYTHEMES

DISSECTION PROBLEMS SOLVED

T. PADMANABHAN

Fascinating as they are, the dissection problems also demand keen thinking from math buffs. Here we solve some of them for you

NOT many readers were successful in solving dissection problems posed in the May 1986 issue of SCIENCE TODAY. Here we discuss answers to those as well as problems presented in the September issue.

The first problem in the May issue was to dissect a regular hexagon into 5 pieces which can be rearranged to form a square. The solution is shown in Fig. 1. The solution is due to Harry

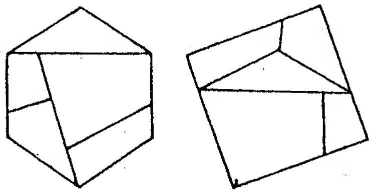


Fig. 1

Lindgram, a patent examiner in Australia who is also one of the world's leading experts in dissection problems. This solution is quite different from the more familiar solution due to Dudeney. Readers who want to match their wits with Lindgram can attempt the problem of Fig. 2 involving a dissection of a

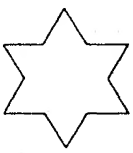


Fig. 2

star into 5 pieces which can be rearranged to form an equilateral triangle.

The second problem was to dissect 2 mitres into 2 congruent squares. This was again achieved by H. Lindgram and the solution is found in Fig. 3 (this

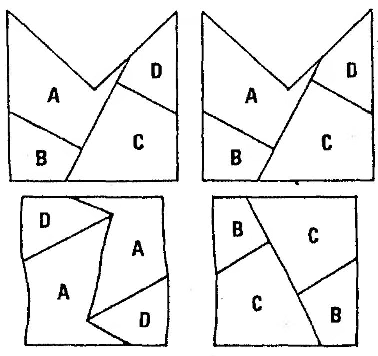


Fig. 3

problem is taken from his wonderful book *Recreational problems in geometric dissections;* Dover, 1972).

The missing feet of the tangrams gentleman are explained in Fig. 4. This

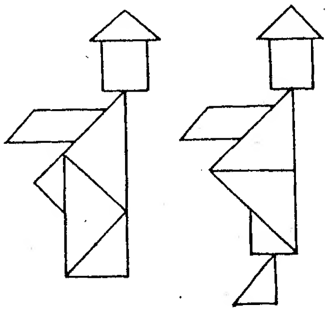


Fig. 4

problem was devised by Dudeney.

The bisection problem given as question 1 at the end of the article can be solved as in Fig. 5.

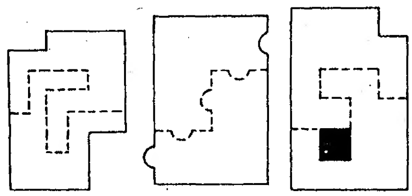


Fig. 5

The second question at the end of the article has the answer: 16 pieces. Clearly, a circle can be cut to the largest number of pieces if each cut intersects the other four cuts. This is achieved in many ways, a prototype being the dissection in Fig. 6.

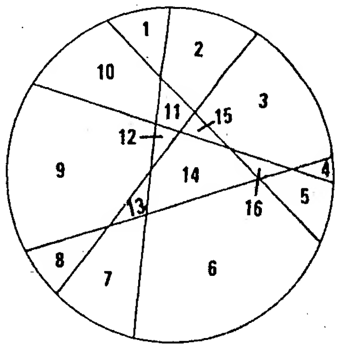


Fig. 6

What if we are allowed to use n lines rather than 5? The general formula giving the maximum number of regions into which a circle can be cut when n lines are used is $(\frac{1}{2}n^2 + \frac{1}{2}n + 1)$. For n=1,2,3,4,5 it gives the correct answers, namely, 2,4,7,11 and 16. There are many ways of deriving this formula but a simple trick is to use a method called 'finite differences'.

The idea of finite differences is as follows. Suppose by simple trial and error you find that the circle can be divided into 1,2,4,7,11 with 0,1,2,3,4 number of cuts respectively. From these numbers, one can produce the 'difference table' shown in Fig. 7. Here,

Numbers	1 2 4 7 11
First differences	1 2 3 4
Second differences	1 1 1
Third differences	0 0

Fig. 7

the number of pieces into which the circle is cut is written in the top row. Each of the lower rows is obtained by taking the difference between adjacent figures in the row just above. The procedure is to be continued until all differences are zeros. If we call the first number in the top most row as A, the first number in the next row as B and so on, the formula generating the nth number is given by

$$A+Bn+C\frac{n(n-1)}{2}+D\frac{n(n-1)(n-2)}{2\times3}+\ldots$$

etc. In the above case, A,B,C are all unity while D and further terms are zero, leading us to the formula given above. This method is useful only when the formula is a simple polynomial. Since there are many problems of this nature, 'finite differences' is a good trick to remember. Here is one more problem which the reader might like to solve: What is the maximum number of triangles that can be made, say, with six lines or, in general, with n lines?

The last question of the May issue can be solved as shown in Fig. 8.

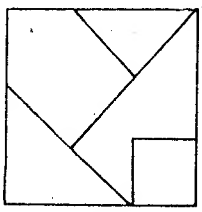


Fig. 8

Unfortunately the figure that appeared on p. 43 of the May issue was somewhat misleading because of a thick line in the smaller square.

Very few readers sent in the correct answer to these dissection problems. Of them Dr. N. Swaminathan of Salem and Mr. S. S. Nayak of Chitradurga have got most of the answers correct. Congrats!

The first problem of the September issue was to find the radius of the largest circle which can be drawn entirely within the black squares of a chess board of size 8 cm. The answer is $\frac{1}{2}\sqrt{10}$ cm. The circle is shown in Fig. 9.

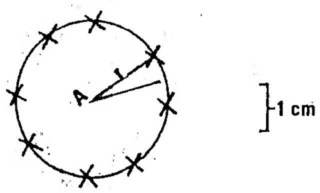


Fig. 9

Correct answers were sent by A. K. Ghose (Varanasi), D. Jana (Calcutta), R. V. Pradhan (Madras), K. S. Mallesh (Mysore) and H. N. Nandeesh (Khetrinagar). Of these, D. Jana gave a detailed argument supporting the solution.

The second problem was to choose a set of 4 different non-zero digits which could not be arranged to make a 4-digit number divisible by 7. Of the 126 different combinations of four-digits which are possible only 3 sets work: 1238, 1389, 2469.

Several readers could find the set 1 2 3 8 but only three people obtained the complete set. They are: A. K. Ghose (Varanasi), D. Jana (Calcutta), H. N. Nandeesh (Khetrinagar).

The (obvious, but nasty!) answer to the third question is 'wrong'. Only three readers—K. S. Mallesh (Mysore), R. V. Pradhan (Madras) and K. A. Anjan (Prakasam)—could see through this word play!

The last question demanded the removal of the minimum number of toothpicks 'killing' all the squares of the 4 × 4 square. The answer is 9 as shown in Fig. 10. To prove this is the minimum, note that the eight shaded cells have no sides in common. Thus, to break these eight squares, atleast 8 toothpicks have to be removed. The same argument applies to the white squares as well. It is therefore preferable to kill the 16 cells with the same eight toothpicks by picking those shared by adjacent cells (so that each removal kills 2 squares). If we do this, none of the removed toothpicks are at the outside border. Therefore, at least one more toothpick must be removed to satisfy the conditions of the problem.

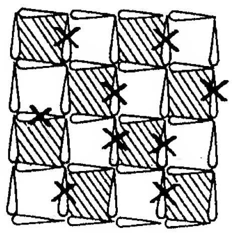


Fig. 10

It is fairly easy to kill all the squares by removing 10 toothpicks and many readers concluded (wrongly) that this is the minimum. Only two readers—N. Swaminathan (Salem) and R. Mohan (Neyveli)—found the 9 toothpick solution.

BRAIN TEASER

SHAIK MAHMOOD

a.	j.	k.	t.	u.
b.	i.	l.	s.	v.
c.	h.	m.	r.	w.
d.	g.	n.	q.	x.
e.	f.	o.	p.	y.

The first 25 letters of the English alphabet are arranged above in a square matrix. Taking the last letter z somewhere outside the arrangement, can you join all the 26 letters by means of eight and only eight straight lines without lifting the pen?

Solution to our February teaser

The overall shape is again a big square to form maximum number of squares with the given match sticks.

And the number of match sticks necessary for n × n squares are equal to $n(n+1)+n(n+1)$

$= 2n(n+1)$

$= 2n^2+2n$

Therefore,

$2n^2+2n = 1300$

Or $2n^2+2n-1300=0$

i.e., $n^2+n-650=0$

i.e., $n^2+26n-25n-650 = 0$

i.e., $n(n+26)-25(n+26) = 0$

i.e., $n=25$ or $n=-26$ So $n=25$

Therefore, the maximum total number of squares $= 25^2 = 625$

PLAYTHEMES

PATH-BREAKING

T. PADMANABHAN

Illustrations: Neeta Karekar

THE town of Königsberg in Russia is built near the river Pregal'a which flows as shown in Fig. 1, and contains the island of Kneipoh. It was the ambition of Leonard Euler (1707-1783)—the most prolific mathematician in history—to stroll along the town of Königsberg crossing each of the bridges once and only once. He couldn't do it but succeeded in proving that it is impossible. Thereby hangs a topologist's tale, that of 'unicursal problems'.

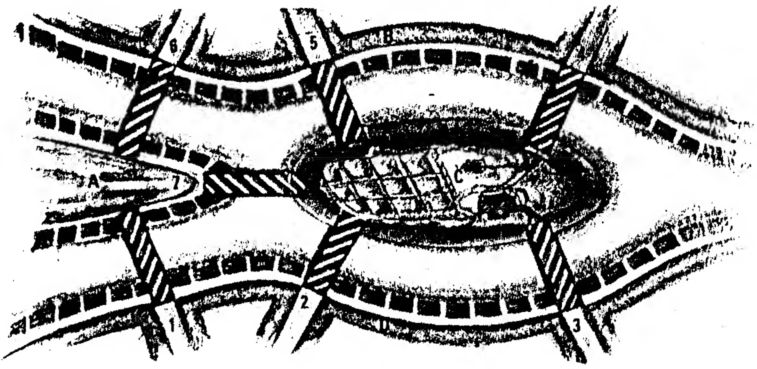


Fig. 1

It is easy to see that the crossing of Königsberg bridges doesn't really depend on the size of the regions or length of the bridges. Taking a highly distorted view of things—which is what topologists specialize in—we can reduce land regions to points and bridges to lines joining these points (see Fig. 2). The problem of Euler's walk reduces to the following question: Is it possible to draw Fig. 2 by one continuous stroke of the pencil, without removing the pencil from the paper or going over the same line twice?

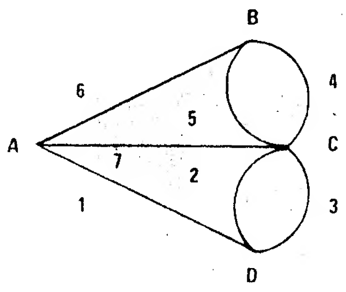


Fig. 2

The answer happens to be no. In arriving at this answer, Euler developed a general set of rules using which one can decide whether any given figure can be drawn unicursally, that is, obeying the restrictions mentioned above. Needless to say, the existence of such simple rules takes away all the charm from unicursal problems. Given any figure, you can immediately decide whether it can be drawn unicursally or not! (If you don't know these rules, try to discover them yourself! I will give these rules when I discuss the answers.) However, recreational mathematicians have thought of modifying the original rules of unicursal problems so as to keep them still challenging. Here are some problems with an unusual twist demanding new pathways. As usual, send the answers to SCIENCE TODAY.

Incidentally, a new bridge has been since constructed in Königsberg making it now possible to realize Euler's dream. Can you guess where the new bridge has to be? Or doesn't it matter?

Moving on to more problems we have: 1. Fig. 3 cannot be drawn unicursally. Let us, therefore, relax the condition that you should not go over the same line more than once. Question: How many continuous strokes, without lifting your pencil from the paper, do you require to draw this design? Everytime you change the direction of your pencil, you should count it as a different stroke, and remember, you can go over the same line more than once!

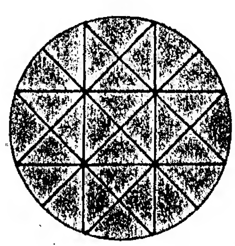


Fig. 3

2. The job of a railway line inspector can be quite tricky. Here, he has to check the 17 railway lines connecting the 12 check points A,B...L (Fig. 4). Obviously he wants to arrange his route so that he can cover all the lines with the minimum amount of travel.

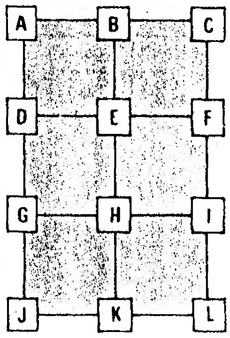


Fig. 4

Assume that the check points are one kilometre apart. What is the minimum distance he has to travel?

3. An icosahedron is a monstrous solid belonging to a class of solids called Platonic solids. (Platonic solid is one with its sides, angles and planes similar and equal.) The icosahedron is bounded by 20 similar equilateral triangles. You can make yourself a model by cutting a cardboard in the shape shown in Fig. 5,

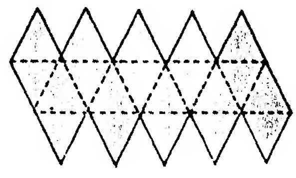


Fig. 5

and folding along the dotted lines; you should get a solid which looks something like Fig. 6.

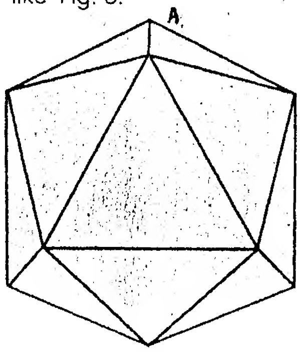


Fig. 6

Now for the question: An ant located at point A plans to walk along each of the edges of the icosahedron. Little thinking convinces the ant that it cannot visit all the edges without visiting some edges more than once. That is bad news; still, can you help the ant to plan the shortest route?

4. Fig. 7 introduces an entirely different kind of challenge. This figure is made of 15 lines connecting 6 vertices of a hexagon. These lines intersect at 13 more points inside the hexagon. So, altogether there are 19 points where the lines meet, marked in the figure by small circles. Think of these circles as 19 towns and the lines connecting them to be roads. You are required to find a continuous path visiting all the 19 towns, starting anywhere. That would have been easy except that the traffic department has declared each of the roads as one-way-paths! In each of the roads the arrow shows the allowed direction of travel; remember that at any small section of the road you must obey the traffic restrictions.

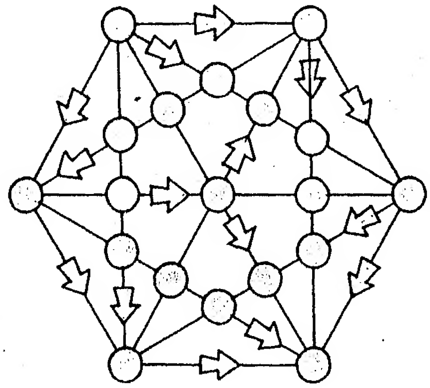


Fig. 7

5. Lastly, you are invited to play billiards on tables which are not exactly normal.

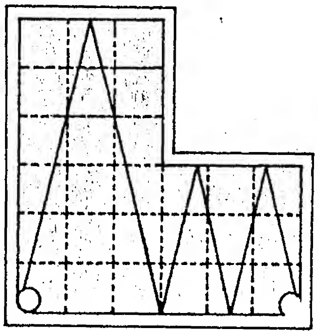


Fig. 8

Fig. 8 shows an L-shaped 'billiard table'. If you want to show off your skill, you can put the ball on to the bottom right hole after 5 bounces, as indicated in the figure. Actually, you can do better! You can pocket the ball after 7 bounces! Can you find that path?

(The grid on the table—shown by dotted lines—is to help you to get the proportions straight. Also note that you are not allowed to bounce a ball at the sharp right angle bend.) ST

MATH COMMUNICATION CONTEST

RESULTS

First Prize: S. Rangarajan,
A 158/1, Jeevan Bima Nagar, D.O.S. Colony,
Bangalore 560 075.

Second Prize: D. Jagan Mohan Rao,
60-16-9, Sidhartha Nagar, Vijayawada
520 010.

Third Prize: G.N.S. Murthy,
IC-703 Bokaro Steel City, Bihar 827 001.

In addition to the above winners, the following contestents deserve special mention for very high quality performance. T. S. Lamba (Kharagpur), G. Rajeev (Trivandrum), A. K. Sahu (Calcutta), R.V. Pradhan (Madras) and Aby Mathew (Trichur).

Congratulations!

We will run a full discussion on the problems in our May issue. Also the regular 'Brain Teaser' column and results of our February teaser are held over.

—Editor

MATH COMMUNICATION

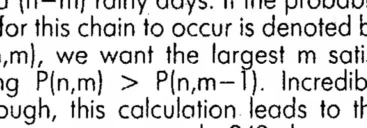
ANSWERS

T. PADMANABHAN

HERE are the answers to the questions of the contest announced in the December 1986 issue of SCIENCE TODAY. Some of the problems allow interesting generalizations which would be discussed in some of the future 'Playthemes' columns.

COME RAIN OR SHINE: 243 days. The 'easy' way to this answer is the following: Let S_k and R_k be the probability for the kth day to be sunny or rainy respectively. Using the conditions of the problem it is easy to show that $S_{k+1} = 1-\frac{1}{2}S_k$, which has the 'steady state' solution $S_{k+1} = S_k = \frac{2}{3}$. Therefore the *mean* number of sunny days is $\frac{2}{3} \times 365$ (or 366) which is about 243. But the question asks for the number of sunny days which occur *most frequently*; which is the *mode* of the distribution! To do this correctly, consider a sequence of n days with m sunny days and (n−m) rainy days. If the probability for this chain to occur is denoted by P(n,m), we want the largest m satisfying P(n,m) > P(n,m−1). Incredibly enough, this calculation leads to the same answer, namely 243 days.

WEIGHT A BIT: If $W_1, W_2 \ldots W_{13}$ form a set of weights satisfying the conditions of the problem, then one can easily see that all Ws must be even or all W_is must be odd. Subtracting the lightest weight from all W_is will lead to a new set (W') which will be even numbers (counting zero as even). If all of them are not zero then we can halve the weights W' and form another set (W''). These operations do not violate the balancing property and can be repeated until we have a mixture of *weights—some odd and some even.* This contradicts the original conclusion, and therefore all (W'_i)s must be zero. (Several readers have sent correct alternative reasonings which of course get full credit.)



SIMILARLY SYMMETRIC: See figure above showing how to divide a (120°, 30°, 30°) triangle in the required manner.

WATCH OUT FOR ANGLES: (a) 143 positions in a 12-hour period. If x and y denote the number of minutes on the face of the clock, indicated by hour hand and minute hand respectively, then one can show that 12x = y + 60N, and 12y = x + 60M where N, M are zeros or any integer from 1 to 11. On solving, we get y = (60/143) × (12M+N). Of the 144 possible values for (12M+N), 2 coincide, giving 143 distinct positions.

(b) Impossible. This condition is very closely satisfied when the time is 3:49:04, 8:10:56, 5:10:56 or 6:49:04. [If x denotes the number of *quarter* revolutions made by the hour hand starting from 12 o'clock, then the conditions can be satisfied only if (12x−x)=11x=A and (720x−x)=719x = B are integers. Clearly A must be multiple of 11 and B must be a multiple of 719 (if A/11 = B/719), making x an integer. The impossibility is proved by explicitly checking for x = 0,1,2,3,4.]

Again, alternate reasonings are possible, but the above two are the shortest.

METHOD IN MADNESS: These are the letters from the bottom row of the standard typewriter. (My apologies, if you think this is unfair! But several *readers did solve it correctly!*)

GOD SAVE THE KING: The black pawn at c7 is the fake one. It has to be replaced by the white king. This remarkable problem (due to Raymond Smullyan) was solved clearly and correctly by only one contestant: Janakiraman (Madras)! (A few readers used completely incorrect reasoning and 'guessed' the final answer, but that is inadmissible.)

In brief the steps in the reasoning are as follows: (1) Assume the king is not at c7. It follows that it can't be the pawns at a7, a6 or b5. Also three white pieces must have been captured on white squares by the black pawns. (2) Since white queen's bishop (on black squares) is also missing, 4 white pieces are off-board. Simple counting shows that, *if c7 pawn is genuine, then the king is substituted by a white piece.* We will now show that this is impossible! (3) Since king cannot be in check, it has to be at d5, e4, g4 or h5. (4) Now consider what could have been the white's and black's last moves. It is easy to see that white's last move was a capture of a black piece at g8 by white queen from f8 [that is, Q(f8) x black piece (g8)] and that black must have moved this particular piece to g8 on *his* last move. (5) Since all white captures are accounted for, this black piece must be a knight coming from f6. (6) Since on the black's move a knight came from f6, none of the pieces on d5, e4, g4, h5 could be white king since these squares are under check from the knight at f6! (7) Therefore, the black pawn at c7 is fake. (The above analysis is highly condensed and only indicates the key steps; I am sure the reader can work out the missing links!)

PRIMARY AVOIDANCE: One such set is N−1001, N−1000,..., N−3, N−2 where N is the product of all prime numbers between 2 and 1001.

FRIDAY THE THIRTEENTH: (a) Suppose 13th December fell on a particular day. Then we can easily compute by how many days the 13th of the next twelve months will go forward. For non-leap year this set is (3, 6, 6, 2, 4, 0, 2, 5, 1, 3, 6, 1) and for leap year this is (3, 6, 0, 3, 5, 1, 3, 6, 2, 4, 0, 2). (That is January 13th will be 3 days ahead compared to December 13th, February will be 6 days ahead etc, etc. Of course, we 'cast out' the 7's.) Since all shifts from 0 to 6 occur, it's clear that 13th of the month occurs on all days at least once. Since no figure occurs more than thrice, Friday the 13th can occur at most thrice in a year—as in 1987!

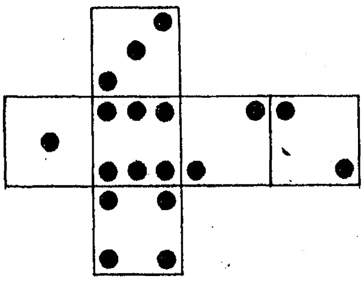
(b) Friday. In our calendar, years divisible by 4 are leap years—except for century years which are leap years only if they are divisible by 400. (Thus 2000 is a leap year but not 1800 or 1900.) It is easy to see that the calendar will repeat itself every 400 years, because 400 year-cycle contains an integral number of weeks. Examining the 1901 to 2301 period one can see that 13th of a month falls on Sunday 687 times; on Monday 685 times; Tuesday 685; Wednesday 687; Thurs-

day 684; *Friday 688* and Saturday 684 times. Thus 13th of a month falls most frequently on Friday. This subtle preference was missed by readers who examined 28 year-cycles.

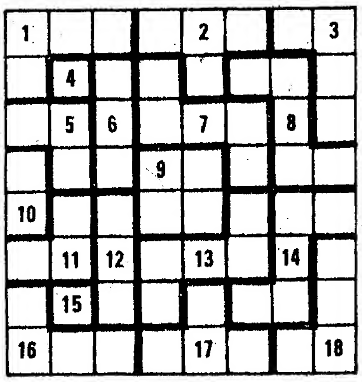
(There is, however, a subtlety: Does 'most frequent' occurrence imply a 'most likely' occurrence, especially when the day on which a particular date falls is a predictable quantity and not a 'random' variable? Readers might like to think about it.)

HOURGLASS FIGURES: Yes. Put the egg to boil and start both hourglasses. When the 7-minute one runs out, turn it over. When the 11-minute hourglass runs out, the other one would have run for 4 minutes. Turn it over and take the egg out when that hourglass runs out.

POINTED MESSAGE: 'It is riches of the mind only that make a man rich and happy' (The arrows with 'bases' indicate space between words.)



ALIE JACTA EST: Two dots. The dice is strange because it has two faces with two dots and none with five (see figure below left).



NO TWINS, PLEASE: The chessboard can be divided into 18 dissimilar pieces. The figure above shows one possible solution. (Many readers have produced other dissections, all of which are correct.)

BONDED LABOUR: It is possible to produce the chain *only if* the finite thickness of the cardboard is used. Using the thickness one can easily 'carve' out the chain from the cardboard. (Several readers have sent actual models prepared this way!)

OUT OF PLACE: The given stanza was a (liberal) English translation of the following French stanza:

Quo j'aime a faire apprendre un numbre utile aux sages!
Immortel Archimede, sublime ingenieur,
Qui de ton jugement peut sonder la valeur?
Pour moi ton problème est de pareils advantages.

This French stanza gives the value of π correctly to 30 places if we use the number of letters in each word as the digits. Virtually every reader thought the stanza was a mnemonic but nobody wondered about a translation. (If you think it was a dirty trick to pull, you will be consoled by the fact that this problem effectively went out of the contest with nobody answering correctly!)

ASKING FOR IT: In a word, the answer is 'yes'!

TICK-TACK-TOE: The correct probabilities are: first player to win (737/1260); second player to win (121/420); draw (8/63). If we forget about the drawn games then first player wins 67 times while the second 33 times.

ONE UP CARDSHIP: To win on the nth draw, the card turned up must be highest of all present, the probability being (1/n). With the 20 cards, the customer can expect to win 10(1+1/2+1/3...1/20) dollars which is about 35.98 dollars. He has to pay 50 dollars. Clearly the game is unfair.

ENDING WELL: 24 zeros. (But if you want to play on words, you can say that, actually, only the last zero is terminating the number!)

FUN WITH MATHS

DIGITAL INVARIANTS

P. K. MUKHERJEE

IF digits of certain numbers are raised to the power of the total number of digits in those numbers and added, the same numbers are generated. These numbers are well-known. For example, by raising *the digits of numbers 153, 370, 371* and 407 to third powers and taking their sums, the original numbers are produced. The sum of the digits of the number 548834 raised to the sixth power similarly generate the same number. The same rule also holds good for 4679307774.

The above are the examples of numbers in which the digits are raised *to the same power*. However, it is also possible to have numbers in which the powers to which the respective digits are raised are in arithmetical progression. Such numbers may be represented by

$N = abc... = a^n+b^{n+1}+c^{n+2}+ ...$

Some examples of two-digit numbers belonging to this category are:

$43=4^2+3^3$ $63=6^2+3^3$ $89=8^1+9^2$

Following are the examples of three-digit numbers:

$135 = 1^1+3^2+5^3$ $175 = 1^1+7^2+5^3$

$518 = 5^1+1^2+8^3$ $598 = 5^1+9^2+8^3$

The examples possible with four-digit numbers are:

$1306=1^1+3^2+0^3+6^4$

$1676=1^1+6^2+7^3+6^4$

$2427=2^1+4^2+2^3+7^4$

Such numbers may be called digital invariants for the operation of raising the digits of these numbers to suitable powers and thereafter, performing the addition operation leaves the numbers unchanged.

It will be interesting to examine if, in addition to the above digital invariants, it is possible to obtain those digital invariants too, in which the powers to which the various digits are raised are the same as the digits themselves. This in effect means obtaining numbers of the form,

$N=abc...=a^a+b^b+c^c+...$

It is found that only two digital invariants satisfying the above requirement are possible, namely,

$3435 = 3^3+4^4+3^3+5^5$

$438579088 =$
$4^4+3^3+8^8+5^5+7^7+9^9+0^0+8^8+8^8$

PLAYTHEMES

COUNTER ATTACK

T. PADMANABHAN

Jump, move, shift, slide or saunter but solve them

IT is difficult to find questions in recreational. mathematics nowadays which are simple but have not been analysed *ad nauseam*. One such rarity is what may be called the 'counter-move' probems. They require quite a bit of ingenuity, judgement and planning—as you will soon discover!

1. Let us begin with the simplest question. Three violet counters and three orange counters are kept on seven cells leaving the centre cell vacant (Fig. 1). The aim is to interchange the

Fig. 1

orange and violet counters by moving them according to some simple rules. The rules are: (i) The orange counters on the left are allowed to move only to the right while the violet ones on the right are allowed to move only to the left. (ii) The counter can jump over a counter of opposite colour provided the cell after the counter jumped over is vacant. An example will make this clear. Starting from the position in Fig.1 suppose you move the orange counter on cell 3 to cell 4. Now the violet counter on cell 5 can 'jump'

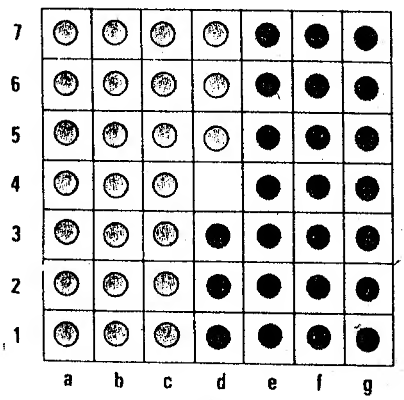


Fig. 2

over the orange in 4 and land up at 3. (Remember that you are not supposed to make an orange counter jump over another orange counter; it has to be of the opposite colour.

Once you have found a way of interchanging the white and black counters, see whether you can do it in the least possible number of moves. There is a solution with just 15 moves with either a 'move' or a 'jump' counting as a single move.

2. Move on now to two dimensions. Fig. 2 shows a 7×7 board with 24 brown and 24 green counters. The central square is left vacant. The green counters can move horizontally to the right or (vertically) from up to down (that is, white can move towards 'east' or 'south'). Similarly the brown pieces can move to the left or move up ('north' and 'west'). Counters can jump over counters of the opposite colour exactly as in the last problem.

Goal: Interchange the green and brown counters in the minimum possible moves which, I believe is 120.

A problem requiring 120 moves for its solution may appear to be incredibly complicated. The charming feature about this problem is that it is ridiculously simple—in fact, the strategy can be worked out mentally—if you go about asking the right questions.

3. If you have solved the last problem, you may think the challenge in Fig. 3 is simpler; after all, there are fewer counters, isn't it? Nevertheless, this is a much tougher task. You should be able to interchange the yellow and blue counters (the rules of movement are the same as in last problem) in 46 moves. Good luck!

4. The last problem is of classical origin belonging to the Victorian era, called the 'Peg Solitaire'. The simplest version is shown in Fig.4. The Solitaire board consists of 33 cells connected as shown in the figure.

To begin the puzzle, place 32 counters on the Solitaire board leaving the central cell (17) vacant. The move consists of jumping over counters and removing the counter which was jumped over (like the way pieces are captured in 'Draughts'). For example,

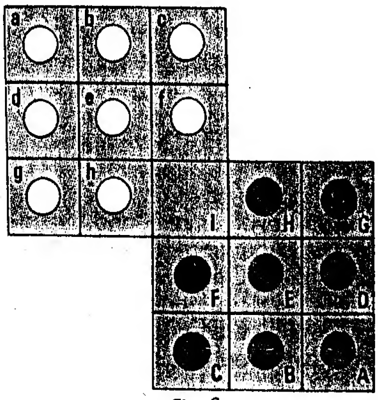


Fig. 3

an allowed opening move would be to move the counter at 15 to the vacant cell 17, jumping over 16 and thereby removing the counter at 16. The goal is to take off all the counters except one in a series of jumps, leaving the last counter in the centre. Remember that the *only* moves which are allowed are jumps; you are forbidden to 'just move' a counter to the adjacent cell.

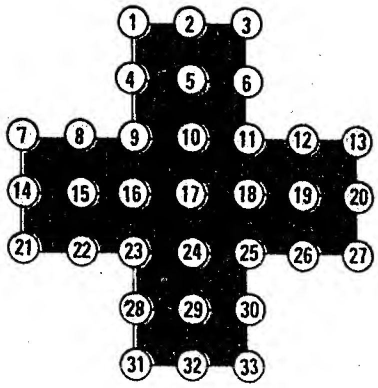


Fig. 4

This is a fairly tough puzzle. The Solitaire board is available commercially in different brand names indicating the ingenuity of the puzzle. If you could get down to 4 or less counters in your first attempt, you are doing well!

Send your answers to SCIENCE TODAY

FUN WITH MATHS

POWER NUMBERS

GOPINATH T.

POWER numbers are of the form $10^m A+B = A+B^n$, where B is the last m digit block, A is the remaining digit block and n is an integer. For example,

m = 1, 36408 $=3640+8^5$
m = 2, 92045 $=920+45^3$
m = 3, 998000999 $=998000+999$

How do we obtain these power numbers?

Let the power numbers be denoted by N. By definition, $N = 10^m A+B$ and $N = A+B^n$. Eliminating A from the above two equations

$$N = \frac{10^m B (B^k -1)}{(10^m -1)} +B... \quad (1)$$

To obtain a power number we have to find the values of B and $k = (n-1)$ to satisfy equation(1) for a particular value of m. Once we have a power number, we can generate many other power numbers for which we need to know two theorems: (1) If y/ (a^k-1), then y/ $(a^{pk}-1)$ where a,x,y,p are integers, and (2) The digital roots of a^k and a^{6y+k} are one and the same where a,x,k are integers. (The 'digital root' of any number is obtained by adding its digits together and treating the resulting number in the same way, until after a certain *number of stages*, the final number consists of only one digit. Thus, the digital root of 656 is 6+5+6 = 17, 1+7 = 8.)

For example, 2^5 = 32, 3+2 = 5
2^{11} = 2048, 2+0+4+8 = 14, 1+4 = 5

Let us take m = 1; the equation (1) reduces to

$$N = \frac{10B (B^k -1)}{9} + B... \quad (2)$$

(i) It is obvious B cannot be 0,1,3, or 6.

(ii) For B=9, the equation (2) is satisfied for all values of n≥2.

For example, 809 = $80+9^3$

(iii) For other digits we try to get a starting value of B and k, so that the digital root of B^k is 1 and use the given theorems to generate other power numbers.

For example, B=8, k=2 gives the power number 568 = $56+8^3$. *Second* theorem gives the general K=6y+k (y=1,2,3,...) For y=1, N=149130808=$14913080+8^9$

First theorem gives the general K = pk (p=2,3...)

For p=2, N=36408 = $3640+8^5$
For p=3, N=2330168 = $233016+8^7$

Some of the examples of generated power numbers: B=44, k=2 gives the power number 86044. First theorem gives general K=pk (p=2,3 ...). For p=2, N=166582044=$1665820+44^5$. Second theorem gives general K=6y+k (y=1,2...). For y=1, N=624365494454044= $6243654944540+44^9$.

Similarly for B=45 and k=2 we have the power number 92045 and using the first theorem we obtain K=pk (p=2,3,4...).

For p=2, N=186392045 = $1863920+45^5$.

BRAIN TEASER

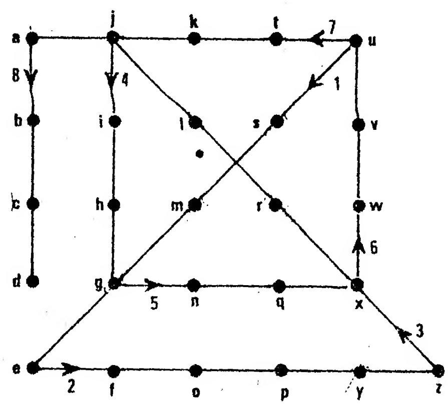
VINAI PRAKASH

There are three women in a party with surnames Gupta, Agrawal and Verma who are a nurse, secretary and a doctor, but *not* respectively. There are also three businessmen in the same party with the same surnames. Can you find out who the doctor is using the following clues?

1. Mr. Agrawal lives in Allahabad.
2. The secretary lives exactly halfway between Allahabad and Naini.
3. Mr. Verma earns exactly Rs.3,700 per month.
4. The secretary's nearest neighbour, one of the businessmen, earns exactly nine times as much as the secretary.
5. The woman with the surname Gupta beats the nurse at rummy.
6. The businessman whose surname is the same as the secretary's lives in Naini.

Solution next month

Solution to March teaser



Congratulations to Archana V., Bangalore, Ravindra Koshti, Bombay, Abhijit Kumar Majumder, Nagpur, R. Dhinagaran, Madras, Vinod S., Bangalore, Joy A. C., Minicoy Island, Soumitra Roy, Calcutta, K. P. Misra, Bhilai, Seema Sen, Durgapur, Santha Srinivasan, Bangalore, H. B. Kantawala, Vadodara, Bijo Varkey T., Trichur, Amalabha Dutta, Calcutta, Vijay Patankar, Bombay, Shah Dinesh H., Gujarat, P. Ramakrishna Iyer, Kerala, Modi Jitendra M., Gujarat, Ketan S. Acharya, Bombay, P. Goukal Gandhi, Gujarat, Ravindra N., Visakhapatnam, Sanjay Kumar Singh, Bihar, Gowri V., Bombay, P. K. Singh, Varanasi, Sandeep Gupta, Saharanpur (UP), Debashish Jana, Calcutta and Ashish Gulhati, New Delhi, who have sent the correct solution to our February teaser.

—Editor

PLAYTHEMES

IT'S ONLY WORDS AGAIN

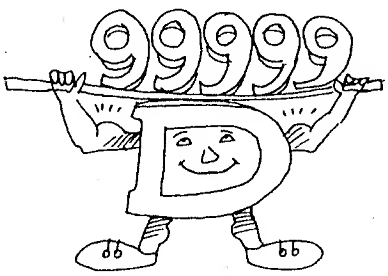
T. PADMANABHAN

The author solves the 'numerous wordy' problems posed in our November '86 issue. How many of you did the same?

Illustrations: Baiju Parthan

THE questions posed in the November 1986 issue produced an interesting set of answers with several good examples of lateral thinking. Let us look at the solutions.

The smallest integers in which the letters a, b, c and d appear for the first time are thousand, billion, octillion and hundred respectively. (Of course, we do not consider the 'and' in numbers like 'one hundred and one'; this was explained in the November column).



Several alternatives were suggested by readers. Many consider 'crore' to be an admissible alternative to 'octillion' (not to mention some others who want to spell lakh as lac!). H.N. Nandesh (Khetrinagar) suggests inclusion of words like double (2), dozen (12) etc. The best lateral thinking was from G. V. Krishna (Visakhapatnam) who pointed out that the problem did not specifically exclude *negative* integers! Since we are looking for smallest integers, negative numbers can play havoc!

The letter which is absent in 0, 1, 2, ... 99 but present in all the numbers 100,... 99999 is d.

★

When the words representing the numbers 0,... 1000 are listed alphabetically the penultimate entry will be 'two hundred two'. Surprisingly enough, only four readers came up with the correct answer; they are: Anshu (Chanda), G. V. Krishna (Visakhapatnam), J. Nandi (Calcutta) and T. Samant (Bombay).

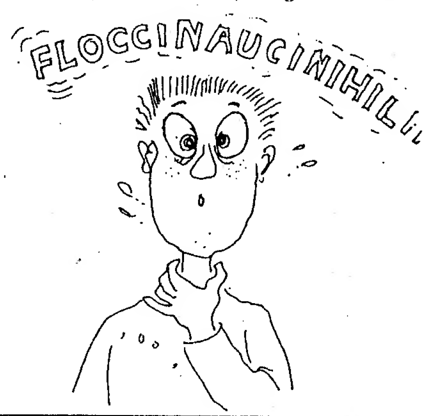
Several readers have given the (wrong) answer 'Two hundred twenty two'. Note that in an alphabetical arrangement this appears *before* 'Two hundred two'.

★

The smallest number which contains a, e, i, o, u and y when spelled out is 'One thousand twenty five'. Correct answers were from Anshu, N. Rajanarayanan (Calicut), B. G. Prashanth (Thirthahalli), Akhilesh Pandey (A.F.M.C., Pune), T. Samant, J. Nandi and T. Ramachandran (Trichur).

★

If hyphenated words are allowed, then one can think of words like 'great-great-...grandfather' which can be arbitrarily long! Several readers have sent alternative answers based on representing numbers by English words.



Among proper, non-hyphenated words, which you can find in a decent dictionary, the word 'FLOCCINAUCINIHILIPILIFICATION' is claimed to be the longest by one reader (J. Nandi of Calcutta). Probably other readers would like to have a go at it!

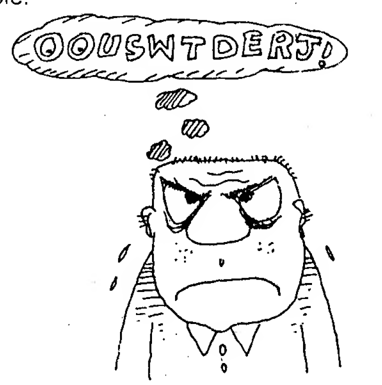
★

This problem is actually based on the first question. Note that 10^3, 10^9, 10^{27}, 10^2, 10^0 are the smallest numbers containing a, b, c, d and e—in that order. So the next number should be the smallest one in which f appears, which is 4. You can, if you want, write 4 as $10^{\log 4}$ using logarithm.

Only 3 readers—J. Nandi, T. Samant and H. N. Nandesh—could crack this one. Congrats!

★

A printing error has made the first part of this problem more irritating than intended. It should have read: 'Rearrange these letters—OOUSWTDNERJ—to spell just one word. Not a proper name or anything unusual'. The answer: JUST ONE WORD. In the printed version an extra E had appeared making the problem unsolvable.



The actual question was to rearrange the letters of NEW DOOR to make one word. The obvious answer is, of course, ONE WORD. The more 'original' solution intended was 'DOORMEN' in which the letter W has been turned upside down to form M (after all, that is 'rearranging', right?). But N. Rajanarayanan (Calicut) has come up with still more originality. He suggests 'MODERN' in which W has been inverted to get M and one O has been put on top of the other O! Congrats!

PLAYTHEMES

ASKING FOR IT

T. PADMANABHAN

Here's a new set of words and figures for you to play with. Sketch, shift and solve them. But remember, your target is to score *exactly* 100

THERE is one puzzle I could never crack—estimating the number of responses to a particular instalment of 'Playthemes'. Though it fluctuates wildly, it seems to obey one rule: Columns which deal with a variety of problems generate better response compared to those dealing with a single theme. So, here comes an assorted pack, testing my theory.

1. What is the area of the largest ellipse that can be inscribed inside a right-angled triangle of sides a and b?

Before you start off with a lot of analytical geometry and calculus and what not, let me tell you that there seems to be a clever way of doing it. Not very rigorous, but definitely clever and a lot of fun.

2. Take a two-digit number. Reverse its digits to form a second number. (If you take 23 as your first number, 32 is the reversed number.) Multiply these two. Now prove that the product cannot be a perfect square except in the trivial case in which the original number has identical digits.

Sure, you can prove this by checking out all the 2-digit numbers. But that would be the dumbest way of doing it.

3. Fig.1 shows a 'word square'. A word square is a square array of letters so that each row (from left to right) and each column (from top to bottom) spells out a meaningful word. Clearly Fig.1 has been scrambled up. Your task is to unscramble it, along these rules: (a) each letter in the array must change places with another letter, and, (b) the only movement allowed is that of a knight in the game of chess.

T	O	A	Y	S	M
S	H	[illegible]	T	T	H
R	E	A	M	S	E
R	S	E	E	M	R
R	S	R	Y	S	T
E	[illegible]	N	E	M	I

Fig. 1

Get busy. It is quite easy if you go about it methodically.

4. What is the minimum number of hits necessary to score *exactly* 100 in this peculiar rifle target shown in Fig.2?

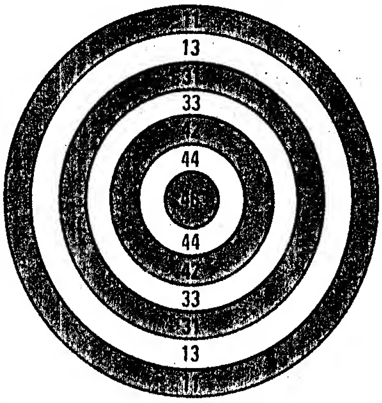


Fig. 2

5. Look at Fig.3(a) and 3(b) which show two views of the same object. In Fig.3(a), we have the front view and Fig.3(b) gives the side view. Task: Sketch the *simplest* three-dimensional object which will conform to these views.

6. Given any obtuse-angled triangle, is it possible to cut it into smaller triangles, all of which are acute?

If you think it is impossible, prove it. If you think it can be done, give a general proof. Remember, the question is not whether some given triangle can be so divided but whether *any* obtuse-angled triangle can be divided into acute-angled triangles.

7. Lewis Carroll (of *Alice in Wonderland* fame) was a prolific inventor of brain teasers and word games. One of the games called Doublets became extremely popular in his days. The idea is to take two appropriate words of same length, then change one to the other by a series of intermediate words each differing by only one letter from the preceding one. (Of course, you can't use proper names etc.) Here is an example of changing PIG to STY: PIG, WIG, WAG, WAY, SAY, STY. The goal is to do it in the smallest number of steps.

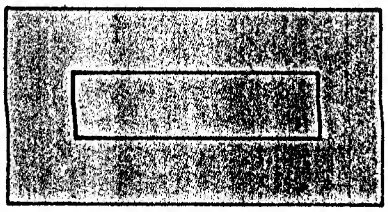


Fig. 3(a)

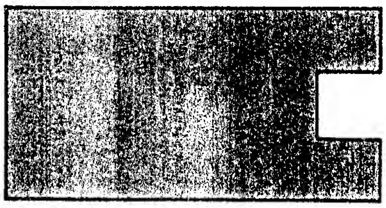
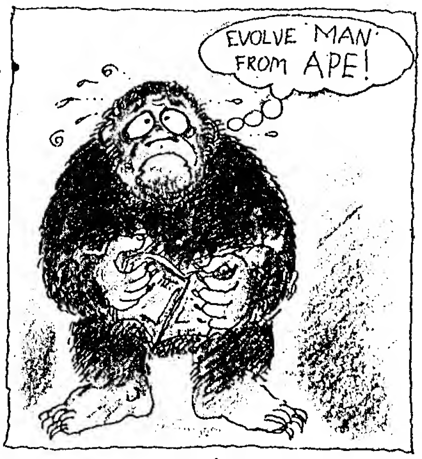


Fig. 3(b)

The English-language magazine, *Vanity Fair* sponsored competitions on Doublets during Carroll's time. Here are five Doublets from the *Vanity Fair* contest: (i) Prove GRASS to be GREEN (ii) Evolve MAN from APE (iii) Change BLUE to PINK (iv) Raise ONE to TWO (v) Make WINTER SUMMER.



BAIJU PARTHAN

Incredibly enough, the solutions suggested by Carroll were not the best ones! Try to solve them along the shortest possible path.

Send your answers to SCIENCE TODAY.

PLAYTHEMES

NUMBER GAMES

T. PADMANABHAN

This time we have 'numerical' problems for you to play with, and solve

THIS time we deal with a set of questions related to numbers. Some are age-old while others are relatively new. Hopefully, each of you will find at least a few of these to be new!

1. Let us begin with a puzzle that dates back to antiquity—the 'game of four'. You are required to express as many positive integers as possible, starting from 1, by using the digit '4' four times—no more, no less—together with standard mathematical symbols. Whenever a number can be expressed in more than one way, the aim of course is to use the minimum possible symbols. For example, you can write $1 = 4/4 + 4-4$; but the same result can be achieved much more elegantly as $1 = 44/44$ or even $1 = 44^{4-4}$.

Numbers from 1 to 10 can be easily expressed using only the symbols for addition, subtraction, multiplication and division (see below). It is usual to grant

$$1=\frac{44}{44}$$

$$2=\frac{4}{4}+\frac{4}{4}$$

$$3=\frac{4+4+4}{4}$$

$$4=4(4-4)+4$$

$$5=\frac{4(4\times4)+4}{4}$$

$$6=4+\frac{4+4}{4}$$

$$7=\frac{44}{4}-4$$

$$8=4+4+4-4$$

$$9=4+4+\frac{4}{4}$$

$$10=\frac{44-4}{4}$$

the symbol '$\sqrt{}$' for square root—used repeatedly, if needed—in order to go further. Using $+, -, \div, \times$ and $\sqrt{}$, it is possible to express all numbers up to 20 except 19! (Can the reader do it?) We may further grant the symbol for factorial '!', (remember that 6! means $6\times5\times4\times3\times2\times1$) and the decimal point '.' (used either as a decimal like 4.4 or as a recurrence symbol, .4 standing for 0.444...) as members of the 'standard symbols set'. With these additions, there are at least 2 ways of expressing 19. Can the reader find them before I discuss the answers in a later issue?

Incidentally, this childish prank of expressing numbers by four 4's has a history dating over a century and attracts papers in mathematics journals even today!

Here are two related questions for you to ponder: (a) It is trivial to express 64 using 4 fours, $(4+4)(4+4)=64$ or 3 fours $4\times4\times4=64$. Challenge: do it with two fours. (b) Can you find a general formula for expressing 19 with four n's where n is some digit in the set (1,2...9)?

2. There are several remarkable properties related to the number 123456789, one of them being the following: Take this number and subtract it from its 'reverse' 987654321. We get 987654321−123456789 = 864197532 which is again made of same digits all scrambled up! The obvious question is: are there other, smaller numbers with this property? Consider a number with all its digits different, and arranged in a descending order; reverse it and subtract it from the original number. The result should be of the same digits, scrambled up.

Incredibly enough, such numbers are rare! There are no such numbers with one, two, five, six or seven digits (Oh yes, I know 0−0=0 but let us not be so smart). For eight digits 98754210 is the unique example (98754210 − 01245789 = 97508421; it is sad to have that zero in the beginning. But if you forbid it, there are no examples with eight digits).

Question: Find the numbers of 3 and 4 digits with this property.

3. Here is yet another classic number puzzle which comes up in different guises: Insert + and − symbols between the sequence 123456789 to make 100.

There are eleven different solutions one of which is $123-4-5-6-7+8-9=100$. Find the solution which uses least number of symbols.

While at it, try to do the same with the reversed sequence of 987654321.

4. Can you have a perfect square (of more than one digit, of course) made of identical digits? You can't? Try proving it!

Next prove that no perfect square of more than one digit can have all its digits odd.

5. Everybody knows that if you have, say, the weights 1 kg, 2 kg, 4 kg, 8 kg, and 16 kg then you can weigh any object with (integral) weight between 1 kg and 31 kg. Here is an equivalent problem on measuring lengths, which is not so well known.

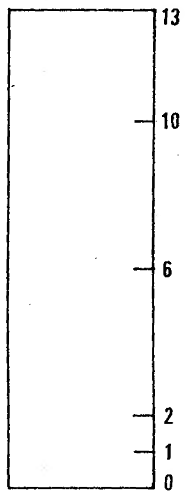


Fig. shown above is, a strange measuring rod 13 inches long. It has only 4 markings at 1,2,6 and 10, but can be used to measure any integral length between 1 and 13. We can measure 1 inch and 2 inches trivially, the mark at 10 allows measurement of 3 inches, 10−6 gives 4 inches, 6−1 gives 5 inches etc.

Task: construct a 12-inch (standard length) rod with minimum number of markings so that all integral lengths from 1 to 12 inches can be measured.

TEASER' TEASERS

B.A. NAIK

THE apparently simple nature of the July brain teaser elicited a tremendous response. SCIENCE TODAY was flooded with hundreds of letters giving 22 different solutions.

Congratulations to all those who have sent the correct answer: 497449. Space constraints do not allow us to publish all the names. However, the first ten entries, were from Sadhana A. Singh, Bombay, Abhijit Sarkar, Bombay, B. V. Raghavan, Salem, Sudhanshu Karandikar, Pune, Deepak Shinde, Sangli, A. Tripathi, Udaipur, D. K. Rakshit, West Bengal, Geetha S., Bangalore, Debasish Jana, Calcutta, and R.D. Goswami, New Delhi.

Among the correct entries, some have solved the teaser with an excellent logical reasoning while others dug out the correct answer in an intelligent manner. (Ranjan Ramchandra, Hubli and T.N. Ram Prasad, Madras). These two found that the PIN Code of Chirimiri from where Debabrata Chatterjee, the author of the teaser hails, matches the answer.

Bapa Dhrangadhara, Bombay, has given some interesting 'square' facts in his answer namely, Chirimiri has nine letters (a square) and 'i' appears 4 times (another square).

Some solutions were interesting examples of lateral thinking. Jharna Hazra from West Bengal and Swapan Mandal from Tripura tried to play on the word 'squares' giving their solutions 648964 and 648064 written as

698969 and

698069

But alas, their solutions are not acceptable, not only because '4' is not written as indicated (9) but also since the third digit '8' is not a prime.

To accomodate 7 squares in a 6-digit number was a challenging job. Do you know the famous puzzle? The population of a town is reduced by three when two fathers and two sons leave it. How? If expressed in an equation 2 + 2 = 3, it looks absurd. However, this appears true provided the trio consists of father, son and grandson! Similarly one gets 3 squares in 49, namely 4, 9 and 49. There can even be 6 squares in a 3-digit number 169: 1, 16, 9, 169, 196, 961 if combinations of digits are also considered. (16+9 and 1+6+9 are also additional squares!) Another interesting 6-digit number is 275625; it gives a square each time its digits are deleted one at a time.

For example,

$$\begin{aligned} 275625 &= (4725)^2 \\ 75625 &= (275)^2 \\ 5625 &= (75)^2 \\ 625 &= (25)^2 \\ 25 &= (5)^2 \end{aligned}$$

Some have sent the answers as 255025 and 255925. These answers are wrong since they do not contain seven squares.

For many the condition 'the third digit is a prime number' was an unsolvable hurdle. A prime number is that which is not divisible by any number except itself and one. 1-digit primes, therefore, are 2, 3, 5 and 7 only. 4, 6, 8 are divisible by 2 and 9 is divisible by 3. Therefore these are not primes. Hence the following solutions were not acceptable as the third digit was not a prime: 164116, 648064, 648964. The digit 'one' is not considered as a prime (otherwise every other number would become composite being divisible by prime number 'one'). Therefore the following solutions were eliminated: 111411, 0931609, 0111601, 0112501.

The number 442144 could have been a good alternative provided combination of digits in any manner, to form squares was allowed. But then there would have been more than 7 squares in the answer. Moreover, in this number the square of the third digit does not occur as the first two digits or last two digits though individually each digit is a square. (The point however is controversial and depends on individual interpretation. The author could have avoided the ambiguous wording.)

Some readers have overlooked apparently trivial conditions while solving the teaser. For example, a PIN code in India is a 6-digit number. Therefore, 5-digit or 7-digit numbers cannot be considered. Hence the following solutions were eliminated: 16416, 2550025, 4421044, 0932509, 0934309, 0935209, 0111601, 0931609, 2550925, 4977649, 4970449, 2552725, 4971349, 0112501.

Even if an answer seems to fulfil all but one condition it is still not acceptable. For instance, 497649. When one is subtracted from this number it is not divisible by 9.

BRAIN TEASER

M.S. TRASI

'Today in class I proved that 1=2', said Kanchan, my granddaughter.

'Huh' I snorted, 'I know all your silly gags. They all boil down to

$1 \times 0 = 2 \times 0$'.

'But this one doesn't—it's different,' she insisted, and proceeded to show me:

Let $x = \sqrt{2}$. Then

$2 = x^2$ .... (1)

On substituting repeatedly, *ad infinitum* for the index 2 on the right-hand side of eq. (1) from the left-hand side of the same equation,

$$\begin{aligned} 2 &= x^2 \\ &= x^{x^2} \\ &= x^{x^{x^2}} = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} \end{aligned}$$

Thus $2 = f(x)$, where $f(x) = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$ and $x = \sqrt{2}$ .... (2)

Now, on squaring both sides of eq.(1), we get

$4 = x^4$ .... (3)

On repeating the same process with this equation as with eq.(1) we have

$$4 = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = f(x)$$

where $x = \sqrt{2}$ .... (4)

On equating eqs. (2) and (4), we obtain

$2 = 4 = f(x) = f(\sqrt{2})$ .... (5)

with f(x) defined as in eq. (2)

Hence $1 = 2$.

Now, my 32-year-old reputation as a mathematician was at stake, but fortunately, to my relief, I could detect the fallacy in a few minutes. Can you?

The answer to our August teaser will be published next month

—Editor

PLAYTHEMES

JUST PLAIN GEOMETRY

T. PADMANABHAN

Illustrations: Baiju Parthan

I'M sure you must have heard of the 'four-colour theorem': 'Four colours are necessary and sufficient to colour any map on a plane so that no two countries with a common border will have the same colour.' It is quite easy to construct maps which *do* require four colours and it is also not too difficult to show that five colours are sufficient. Filling the gap between five and four entailed considerable effort. In 1976, W. Haken and K.Appel of the University of Illinois proved the four-colour theorem using a computer program which ran for 1,200 hours!

We begin this instalment with related, but considerably simpler questions:

1. Beginners often 'prove' the four-colour theorem by proving an easier result: 'It is impossible to draw a map of five regions in such a way that each region is adjacent to the other four.' (Two regions are adjacent if they share a common border; two regions which meet at just one point will not be considered adjacent.)

2. If you put restrictions on the kind of maps under discussion, then interesting results can be proved. Consider, for example, a large region in a plane divided into countries by straight lines (Fig. 1). If the region was rectangular and the lines were all horizontal and vertical, we need only two colours. (Think of a chess board.) Now prove that even for maps like the one in Fig. 1, two colours are enough. (You can, of course, easily verify that Fig. 1 can be coloured with two colours; that is *not* the task. You have to show that *any* map made of straight lines require only two colours.)

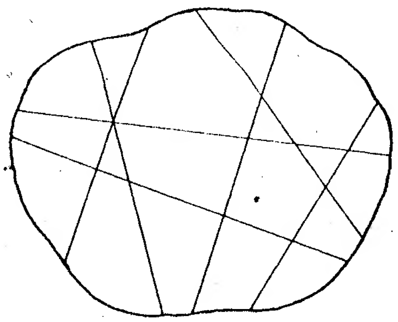


Fig. 1

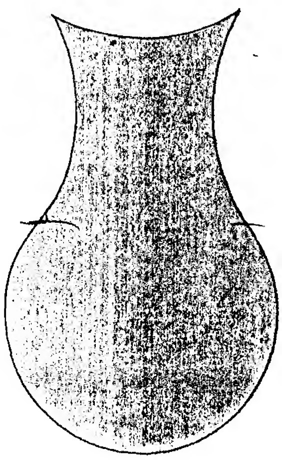


Fig.2

The result is actually more general. *Any map drawn with (a) 'endless' lines* which cut across the entire map, and (b) 'closed' lines which lie on the map as simple closed curves, can be coloured with two colours.

3. Fig. 2 shows the cross-section of an oriental vase. The lower part of the vase is three-fourths of the circumference of a circle with a diameter, say, 1 metre. The upper part is bounded by three quarter-arcs of a circle of the same size.

Question: What is the size of a square equal in area to this figure?

Hint: This question can be answered mentally without doing any calculation.

4. With three lines of equal length it is easy to divide a circle into three equal parts, as shown in Fig. 3. The task here is to divide the circle into four equal parts using three lines of equal length. The lines do not have to be straight, but they must not cross.

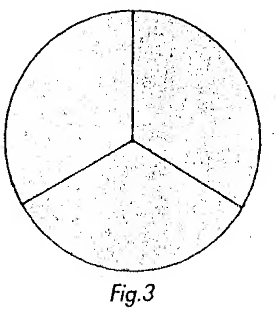


Fig.3

5. This delightful problem was coined by H.E. Dudney, who gave it in the form of a story: A lady, more rich than wise, inherited a ruby brooch from her mother. The brooch had the general shape shown in Fig. 4. Every once in a while she used to make sure that all the stones in the brooch were intact by the following—rather eccentric—method of counting: she would start from the centre and count up one line along the outside rim and down the next line and make sure that there were eight rubies. She knew that there were eight along each such route when she got it from her dear mother and believed that this form of counting would let her know if any of the rubies was missing.

One day her uncle—who was visiting—looked at the brooch and was quite aghast. 'Oh dear', he said, 'I knew for sure this brooch originally contained forty-five rubies and now there are only forty-one. Your jeweller, while repairing the brooch sometime, has stolen four rubies and reset *as small a number of rubies* as possible, keeping your count of eight intact.' Later investigation proved the uncle's observation to be right and the jeweller a thief.

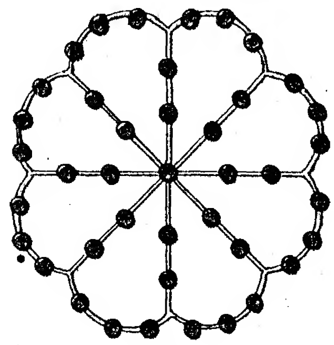


Fig.4

Question: Fig. 4 shows the brooch as the uncle saw it after the theft. It has 41 rubies and satisfies the 'count of eight' criterion. What was the original configuration? Which 4 rubies were stolen and how many were reset?

PLAYTHEMES

ANSWER SQUARED

T. PADMANABHAN

THE problems in the February '87 issue were probably the easiest ones which have appeared so far in Playthemes! Several readers have sent correct solutions. Let us look at the answers.

The first problem was to construct a 3×3 magic square with the digit 8 in the upper middle cell. Other conditions were left open-ended. This task is im-

15	8	13
10	12	14
11	16	9

Fig. 1

possible if we are permitted to use only the integers 1, 2..., 9. If other integers are allowed there are several possible solutions. We can try to be as close as possible to the conventional 3×3 square by demanding the integers to be consecutive. It is trivial to make such a square. Note that a magic square remains 'magic' even when you add a constant number to all the cells. The usual 3×3 square (see Fig. 2, p. 76 of the February issue) has '1' in the top middle cell. So by adding '7' to all the cells we can get a magic square, made of consecutive integers, with 8 at the top (see Fig. 1). The magic constant is now 36. On the other hand, one can try to form a magic square with the usual constant (15) with 8 on top. This is impossible with integers but can be done easily with fractions. One feasible answer is shown in Fig. 2.

Several readers have sent correct answers. Detailed analysis was provided by D. Jana (Calcutta), N. Sekar Kar (Howrah), Santha Srinivasan (Bangalore) and R.V. Pradhan (Madras).

4½	8	2½
3	5	7
7½	2	5½

Fig. 2

The second problem was to construct magic squares of multiplication and division. If no constraints were imposed,

a^8	a^1	a^6
a^3	a^5	a^7
a^4	a^9	a^2

Fig. 3

then every addition magic square can be used to produce infinite number of magic squares of multiplication. Consider the 3×3 square in Fig. 3 where 'a' is some positive integer. Since $a^m \times a^n = a^{m+n}$, the product of digits along the horizontal, vertical or diagonal will be the same: a^{15}!

12	1	18
9	6	4
2	36	3

Fig. 4

Though we have succeeded in forming multiplication squares these are far from the smallest ones that can be constructed. The 3×3 magic multiplication square with the smallest constant is shown in Fig. 4 (which multiplies to 216), assuming all the cells contain different digits.

The subtraction square discussed in the February issue can be converted into a division square by the same trick used in Fig. 3. It is also not difficult to

3	1	2
9	6	4
18	36	12

Fig. 5

rearrange the digits of Fig. 4 to obtain a division square (see Fig. 5). This has the smallest possible division constant (6) for a 3×3 square with different digits.

Several readers attempted the generalizations to multiplication and division squares. Solutions of the form in Fig. 3 were constructed by D. Jana, A.K. Sahu (Calcutta), Santha Srinivasan, R.V. Pradhan, and R. Dhinagaran (Madras). Of these, A.K. Sahu has given the minimal multiplication squares of 3×3, 4×4 and 5×5 orders. Congrats!

The solution to the third problem is shown in Fig. 6. One strip is left uncut

1 2 3 4 5 6 7

3 4 5 6 7 | 1 2

5 6 7 | 1 2 3 4

7 | 1 2 3 4 5 6

2 3 4 5 6 7 | 1

4 5 6 7 | 1 2 3

6 7 | 1 2 3 4 5

Fig. 6

and the other six strips are cut at one place each. These 13 pieces can be rearranged to form a magic square as shown in Fig. 6. I'm afraid several readers didn't quite understand what was required of them; probably I should have phrased it differently. Correct answers were sent by R.V. Pradhan, Santha Srinivasan, N. Sekar Kar, and Shah Dinesh (Gujarat).

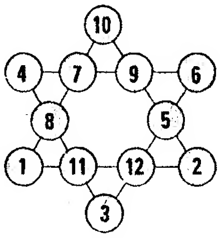


Fig. 7

The last problem was correctly solved by almost all readers who sent the solution. There are several ways of arranging the numbers in the hexagram but the most symmetric solution is probably the one shown in Fig. 7.

The magic square problems were adapted from H.E. Dudney's *Amusements in Mathematics* (Dover, 1970).

In the first question of Playthemes (SCIENCE TODAY, September 1987, p.69) the number 5 was expressed as $5 = \frac{4(4\times4)+4}{4}$. *The correct equation is* $5 = \frac{(4\times4)+4}{4}$. *The error is regretted.*

—Editor

The problem posed in the August Brain Teaser was apparently successful in tickling the readers' minds. This could be deduced from the many and varied solutions we received at SCIENCE TODAY *some of which we reproduce below.*

It is known that the magnetic intensity (or potential) at a point 'P' due to a bar magnet is given by

$$\frac{1}{4\pi\mu_0}\left[\frac{\mu\cos\theta}{r^2-l^2\cos^2\theta}\right]$$

where μ is the magnetic moment, r is the distance between the point 'P' and the centre of the magnet and θ the angle of inclination of the straight line joining 'P' and 'O'.

If $\theta=90$ then potential = 0

That means the point will neither feel attraction nor repulsion. I suggest this principle to unravel the problem. Let the two rods be A and B. Now keep the two as shown. If A is not a magnet then the force of attraction experienced by it will be minimum. (Conversely if A is a magnet then the force should be maximum.) This can be further confirmed by interchanging the rods. It is actually by this principle that we identify the magnet among the two bars.

R. DHINAGARAN
Madras

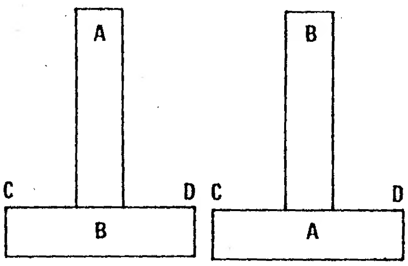


Fig. 1 *Fig. 2*

Mr. Dhinagaran's answer is similar to that given by readers whose names appear in the box. If bars A and B are placed as shown, assuming A is a magnet, the force of attraction between A and B will be same at all points along segment CD (Fig. 1), whereas in Fig. 2 the ordinary bar will not experience any attraction since the net force at the centre of the magnet is zero.

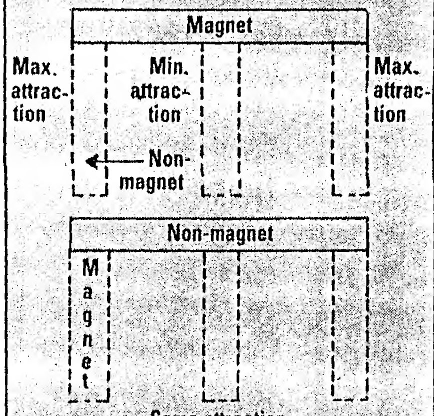
The man first takes one of the two bars and magnetically induces the second bar (by the single touch method of magnetic induction).

He then tests the two bars (knowing the one which has been induced) for repulsion. If there is repulsion, then the magnetically induced bar is the ordinary bar while the other one is a magnet.

If, however, there is no repulsion he repeats the procedure of magnetic induction with the second bar, that is, he magnetically induces the second bar.

Suppose the first bar the man takes is the magnet. Now by taking the other rod as shown in the figure below he finds out the attractive force to be maximum at the two ends and minimum at the centre. In case he takes the non-magnetic bar and tests it with the magnet he will find that the attraction is same at all points. It is interesting to note that if in place of the ordinary and magnetic bars we had a charged metallic ball and an uncharged one—there is no way to find out as to which one has the charge. Can the readers explain why?

PREM PRAKASH SINGH
Bombay



Several readers including B. S. Venkatesh, Bangalore, Sunita Kothari, New Delhi, A. C. Phadke, Bombay, B. V. Raghavan, Salem, Raman Chopra, Chandigarh, Rahul Bose, New Delhi, Apu Sivadas, Calicut, S. N. Sharma, Calcutta, Sanjay Sethi, Jhansi and Nabendu Dev, Jaintia Hills (Meghalaya) have sent this answer.

Now, there has to be repulsion since the second bar has been magnetized and thus he is able to find out which of the two bars is the magnet.

J. T. MISTRY
Bombay

This answer was also sent by Madhav Joshi, Bombay, and Girish Menon, Bombay.

It is quite possible to magnetize the non-magnetic bar and thus test which bar is the magnet. But the process of inducing magnetism is a slow one and one will have to stroke the ordinary bar with the magnet a large number of times.

If the bar magnet is placed on the bald head of the man (assumed to be smooth, slippery and curved) it always gets oriented in the north-south direction whereas in the case of the ordinary bar no such fixed orientation results.

PARTHA CHOUDHURY
IIT, Kharagpur

Others who wanted to help the man with this solution where A. K. Gorai, Chandrapur and Bipin Chandra Chougule, Pune.

Theoretically this solution is feasible, But this method will be rather difficult to apply in practice since, for the magnet to orient itself north-south on the man's head, the latter will have to be smooth, curved and touched at only one point by the bar, in order to ensure minimum friction.

The man must drop one of the rods to the ground very hard. There are two possibilities now.

1) The two rods cease to attract each other: In this case, the rod that he dropped was the magnetized one.

2) The two rods still attract each other, then the rod he dropped was the ordinary rod.

ASHISH GULHATI
Delhi

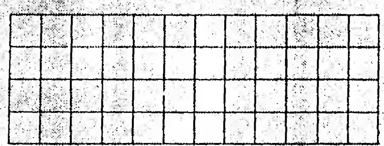
Alternately, if the bar is brittle, it will break on being thrown to the ground and by checking for attraction (or repulsion) between the broken pieces of one bar, one can determine which is the magnet. (If the bar flung to the ground is the magnet, each of the pieces it breaks into will also be a magnet. If the bar thrown is the ordinary one there will be no attraction or repulsion between the broken pieces.)

There are large clusters of atoms in a magnet, called domains. All atom magnets in one domain are lined up in the same direction. A permanent magnet can lose its magnetism if it is hit hard (but this again is a slow process; the magnet may not be completely demagnetized in one throw). This is because the jolt which the magnet receives shakes the atoms in the domains from their aligned direction. Heating a magnet above a certain temperature called its Curie temperature can also demagnetize it.

BRAIN TEASER

K. ANJAN

Matchsticks are used to arrange the 48 squares shown here. You are allowed to rearrange the squares without changing the number of matchsticks used. How many more squares can you develop?



PLAYTHEMES

STRIP-TEASERS

T. PADMANABHAN

EVERY brain teaser fan knows about the Möbius strip—that strange piece which can be constructed by pasting the ends of a twisted strip of paper—which has only one side. By changing the rules of the game slightly we can construct several intriguing brain teaser strips. Here are some of them.

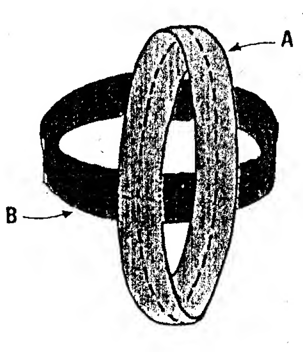


Fig. 1

1. The double strip in Fig. 1 is constructed by joining 2 paper strips, one of size 1 cm x 10 cm and the other 1 cm x 9 cm. Call the first one A and the second B. Before being attached together, B is just a cylinder while A has two sides, two edges and two half twists. The final form in Fig. 1 has 2 sides but only one edge.

Now we mutilate it by cutting along the dotted line. Tell me, what will be the result? That is, give the number of sides, edges and twists of the resulting figure(s).

2. If a simple strip of paper is joined without twist (making a cylinder) and then cut lengthwise along the middle it will come apart as two similar pieces.

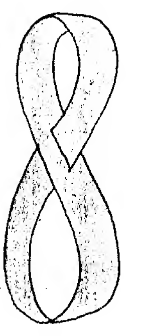


Fig. 2

The Mobius strip—made by giving half a twist before pasting—will remain as one piece under such a treatment.

Suppose we make a slit in the strip before joining and push one end through the slit and join we will get a figure like the one in Fig. 2. Now suppose we cut through the middle along the length of the paper, making the slit part of our cut, what will be the result? Give number of loops, twists etc.

3. The following problem is a creation of Stephan Barr, the American puzzler. He has a dressing gown with a long cloth belt the ends of which are cut at 45° as shown in Fig. 3. He would like to pack it by rolling it as neatly as possible, that is, he wants to fold it first so that he obtains a rectangle of uniform thickness which can then be rolled compactly. What is the simplest way of doing this?

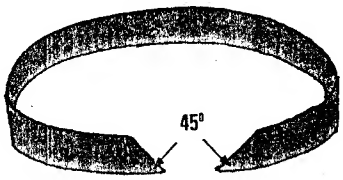


Fig. 3

4. While we are dabbling with topological creatures like Möbius strips we might as well discuss an age-old problem of a torus. Fig. 4 shows a torus with a hole in it. An incredible topological fact is that such a torus can be 'inverted'—that is, turned inside out—by stretching the surface without tearing. (Can you see how?)

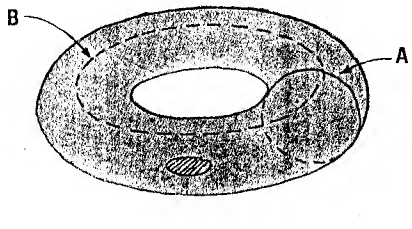


Fig. 4

Once you succeed in inverting the torus, worry about the following question. In Fig. 4 I have drawn 2 rings. Ring A is on the outside and ring B on the inside. These rings are clearly linked. But on inverting the torus we will end up with a configuration shown in Fig. 5.

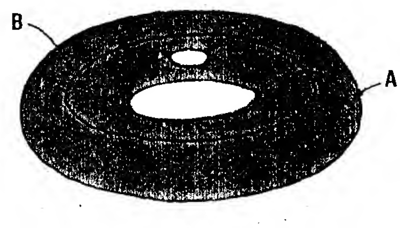


Fig. 5

Now the rings have become unlinked! Doesn't it violate one's intuition?

5. Incidentally, what initial proportion must a torus have so that the inverted one has the same proportion? Or is it not possible at all?

6. A topological party trick is based on a long cord and a door key. Arrange the keyhole, cord and key as shown in Fig. 6(c). Tie the ends of the cord A and B to the back of a chair or something. Allow plenty of slack in the cord. The problem is to manipulate the key and the cord so that the key is moved from P to Q leaving the cord looped as it is. It is tougher than you think.

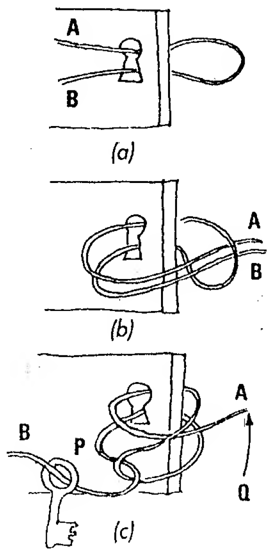


Fig. 6

Send the solutions to SCIENCE TODAY.

PLAYTHEMES

NOW FOR THE PATHS

T. PADMANABHAN

TO discuss the answers to the questions of April 1987, it is necessary to introduce a little bit of terminology describing paths. We will call each point from which lines emerge (like A, B, C, D in Fig. 1) as a 'vertex' and the number of lines which emerge from each vertex as 'the degree of the vertex'. Thus, in Fig. 1, B is of

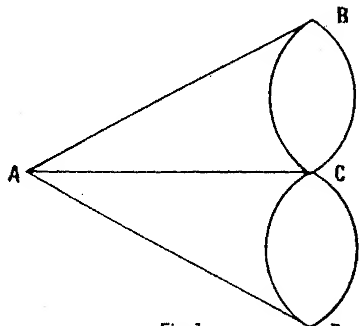


Fig.1

degree 3 (since 3 lines emerge from B), while vertex C is of degree 5. A vertex is called 'odd' or 'even' depending on whether its degree is odd or even.

Simple answers can be given to several interesting questions based on the degrees of the vertices. Suppose every vertex of a figure is even, then that figure can be traced in a single continuous stroke of the pencil, traversing each line once and only once and getting back to the starting point. If there are only 2 odd vertices, we can still trace the graph with a single stroke crossing each line once and only once—but we can't come back to where we started. In fact, it is necessary to start at one odd vertex and end at the other. If there are more than 2 odd vertices then we cannot achieve this single-continuous-stroke business. (What if the figure has just one odd vertex? Sorry, it can't happen. I will leave it to the readers to prove.)

The Königsberg problem, depicted in Fig. 1, has all the 4 vertices odd. Hence it can't be traced. If a bridge is built between any two islands it would make those two vertices even, leaving just 2(other) odd vertices. Now a path can be constructed starting at one of them, traversing each bridge once and only once and ending at the other. (This, of course, won't let you come back to the starting point.)

Incidentally, this is quite different from the task of constructing a path which will visit each *vertex* once and only once (with no restrictions on how many times the *lines* are crossed). Incredibly enough, there is no simple answer to this problem for a general figure.

Now for the answers: The first problem can be traced in 12 continuous strokes as shown in Fig. 2. Starting at A, one can form the star in eight strokes and come back to A; then one stroke round the circle to B, one stroke to C, one round the circle to D and one final stroke to E solves the problem—12 strokes in all with partial overlap. Correct answers were sent by R.V. Pradhan (Madras), M.G. Timol (Valsad) and C.M. Jirati (Bhilai). Several people (like D. Jana, Calcutta) reached a 13-stroke

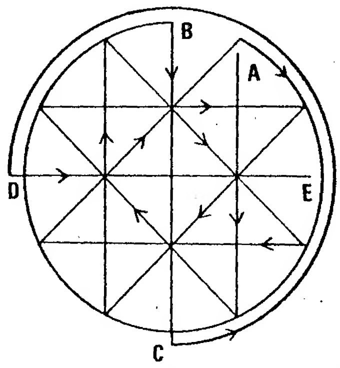


Fig.2

solution by starting at the wrong place!

The second problem can be solved by travelling just 19 km. One possible route is: B A D G D E F I F C B E H K L I H G J K (see Fig. 4 of Playthemes, April '87 page 75). The only portions of the line travelled over twice are the two sections D to G and F to I. Several alternative routes exist but none shorter. Correct answers came from C.M. Jirati, K. Suresh Kumar (Trivandrum), Shankaranarayanan (Alwaye) and K. Venkatesh Arvind of Madras. Many arrived at a 20 km route by making the inspector walk along DG and (say) KI twice. A more incredible (and interesting) attempt was to persuade our man to cross diagonally from K to I bringing in some irrationality into the problem. A couple of readers decided that the inspector is interested in visiting each *check point* rather than in inspecting every line and ended up 'solving' a much easier problem.

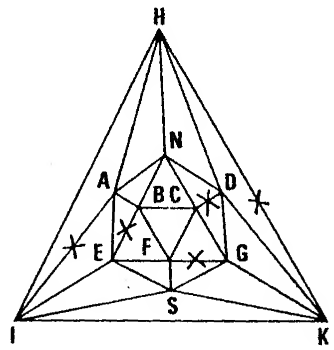


Fig.3

The icosahedron puzzle can be solved with the help of Fig. 3 in which we have flattened out the solid. The 18 edges visible in the original diagram (Fig. 6 of Playthemes, April '87) form the hexagon N A E S G D. Rest of the figure is the 'backside'. (19 triangles are clearly there and the 20th one is H I K.) To find the shortest route, proceed as follows. I have crossed out 5 edges in the figure so that all the vertices are now even except N and S. Clearly, one can devise a path from N to S covering each edge once and only once. The crossed-out edges can be incorporated by going over each of them twice. If each edge is one unit, the ant has to travel 25 units plus 2 × 5 units—a total of 35 units. Here is one such route: N to H, I, A, I, K, H, K, S, I, E, S, G, F, G, K, D,

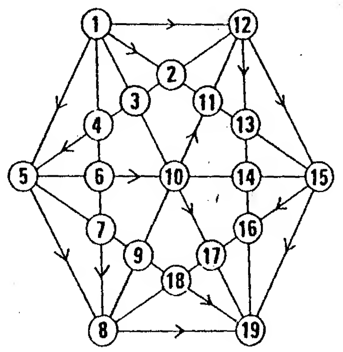


Fig.4

C, D, H, A, N, B, E, B, A, E, F, B, C, G, D, N, C, F, S. This probably was the toughest problem. Only 2 readers—R.V. Pradhan and Suresh Kumar—solved it right. Aditya Choudhary of IIT, Kharagpur came up with a 36-unit solution.

The one-way traffic problem has the solution shown in Fig. 4. Note that he must start at 1 and end at 19. Correct answers came from Suresh Kumar, R.V. Pradhan, C.M. Jirati, D. Jana, Shankaranarayanan, Venkatesh Arvind and M.G. Timol. The last problem has the answer shown in Fig. 5. This was solved correctly by D. Jana and Lakshminarayanan (Madras). Several readers went wrong in not making sure that the 'angle of incidence is equal to the angle of reflection, in bouncing the ball.

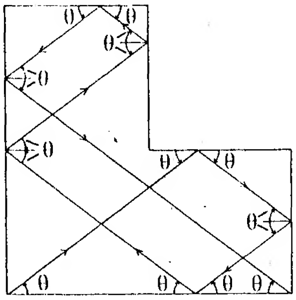


Fig.5

The minimal path problems are due to H.E. Dudney and the last 2 problems are the creations of Ivan Moscovich, the puzzler from Tel Aviv.

SPOTTING FALLACY

B. A. NAIK

THE Brain Teaser posed in our September '87 issue elicited an interesting response. In attempting to identify the fallacy in the granddaughter's statement, the readers themselves posed a few teasers.

One reader concluded that '2=8' can also be proved as he assumed that $8=x^8$ which is not true in the present context. It appears that he overlooked the fact that $n=x^n$ has only *two real distinct* roots: n=2 and n=4.

Apparently, the phrase, 'substituting repeatedly ad infinitum', and equating the expressions on the right hand side created doubts about the equality in the minds of many readers, as they found the last exponent in the expression

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

as different. However, no sooner do we think of an exponent for this expression after, say, n terms than it ceases to be an expression containing 'infinite' terms.

The mathematics of infinity is quite an absorbing and debated subject among amateurs and at times among mathematicians too. However one representative letter expressing the opinion of many readers is presented here. Kapilesh Srivastava writes:

The fallacy is created by the term *ad infinitum*. By infinitely substituting we get

$$2=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \quad \ldots (1)$$

But we must not forget that in (1) the last exponent is actually x^2,

$$\text{that is, } 2=x^{x^{\cdot^{\cdot^{x^{2}}}}} \quad \ldots (2)$$

Same is the case with $4=x^4$ which on substituting infinitely becomes

$$4=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \quad \ldots (3)$$

But in (3) the last exponent is actually x^4,

$$\text{that is, } 4=x^{x^{\cdot^{\cdot^{x^{4}}}}} \quad \ldots (4)$$

You have taken (1)=(3)
but actually, (1)=(2)
and (3)=(4).
Clearly (2)≠(4).

Since the last exponent of (1) and (3) are not given they seem to be equal. This is the fallacy in the statement 1=2.

Shakuntala Devi, among others, has explained the fallacy in the argument '1=2' in her book *Figuring: The Joy of Numbers*, which deals with the violation of the mathematical rule that division by zero is not allowed.

Let a=b

Then multiplying by b

$ab=b^2$.

Subtracting a^2

$ab-a^2=b^2-a^2$
$a(b-a)=(b+a)(b-a)$
$a=b+a$
$a=a+a=2a$
Hence, 1=2.

BRAIN TEASER

P. K. MUKHERJEE

There exists a square number which can be expressed as the product of four consecutive odd integers. Can you figure out that number?

Solution to September teaser

The equations $2=x^2$ and $4=x^4$, where

$x=\sqrt{2}=2^{1/2}$

can be written in the general form

$n=x^n=2^{n/2}\ldots$

This is known as a transcendental equation; it has only *two real distinct roots* n=2 and 4. The function

$$f(x)=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \quad \ldots (a)$$

obtained from (a) by repeated substitution for n is therefore two-valued. The fallacy lies in having equated the two distinct values of f(x); this is just like writing

$3=\sqrt{9}$
$-3=\sqrt{9}$
Therefore 3=−3.

PLAYTHEMES

OLYMPIAD STUFF

T. PADMANABHAN

Here's an opportunity for you to pit your wits against Olympiad stuff—complete with counterfeit coins, pet monkeys and coconuts

ONE of the most prestigious contests for high school children in the USSR is the mathematical Olympiad. Here is a selection of some of the problems which had appeared in this contest. These are slightly more difficult than what you normally encounter in this column, even though I have picked the easier ones. But then, don't let that deter you!

*

1. You are given N coins, all of which look alike. One of them is counterfeit having a weight different from the rest. You don't know whether it is lighter or heavier. What is the least number of weighings in a pan balance needed to isolate the counterfeit? Weighing involves placing coins on the pans of the balance and noting how the scales dip. No calibrated weights are given. In case an abstract number like N scares you, try it first for, say, 1,000 coins.

*

2. 200 students are positioned in 10 rows, each containing 20 students. From each of the 20 columns the shortest student is selected, thereby getting a set of 20 students. Next, the tallest among these 20 'short' students is selected and he is marked 'A'. All the students return to their initial positions. We now pick the tallest student in each row and obtain from these 10 'tall' students, the shortest one. He is labelled 'B'.

Now tell me, who is taller—A or B?

*

3. Look at the number triangle formed here. Each number is formed by adding three numbers of the previous row: the number immediately above and the numbers immediately to the right and left—all in the previous row. If no number appears in some of these locations we treat it as zero.

Prove that every row beginning with

1
111
12321
1367631
....

the third one contains at least one even number.

*

4. Find all integers N which are divisible by all integers not exceeding $\sqrt{N}$.

*

5. Lastly, here is a problem which has appeared in many guises—known as 'The monkey and the coconuts'. An island has five men and their pet monkey. One afternoon they gathered a large pile of coconuts and decided to divide it equally among themselves the next morning. That night the first man decided to help himself to his share. Dividing the coconuts into five equal parts, he found that there was one left over and gave it to the monkey. He hid his one-fifth share and put the rest back as a single pile. As you would have guessed, the second man also woke up, got the same idea, went to the pile, divided it into five equal parts, found one left over, gave it to the monkey and hid his share. All the five men repeated the procedure, each time giving one left-over coconut to the monkey. Next morning they went to the considerably diminished pile of coconuts and divided it among themselves equally. Again they found that there was one coconut left over.

What is the least number of coconuts the original pile could have contained?

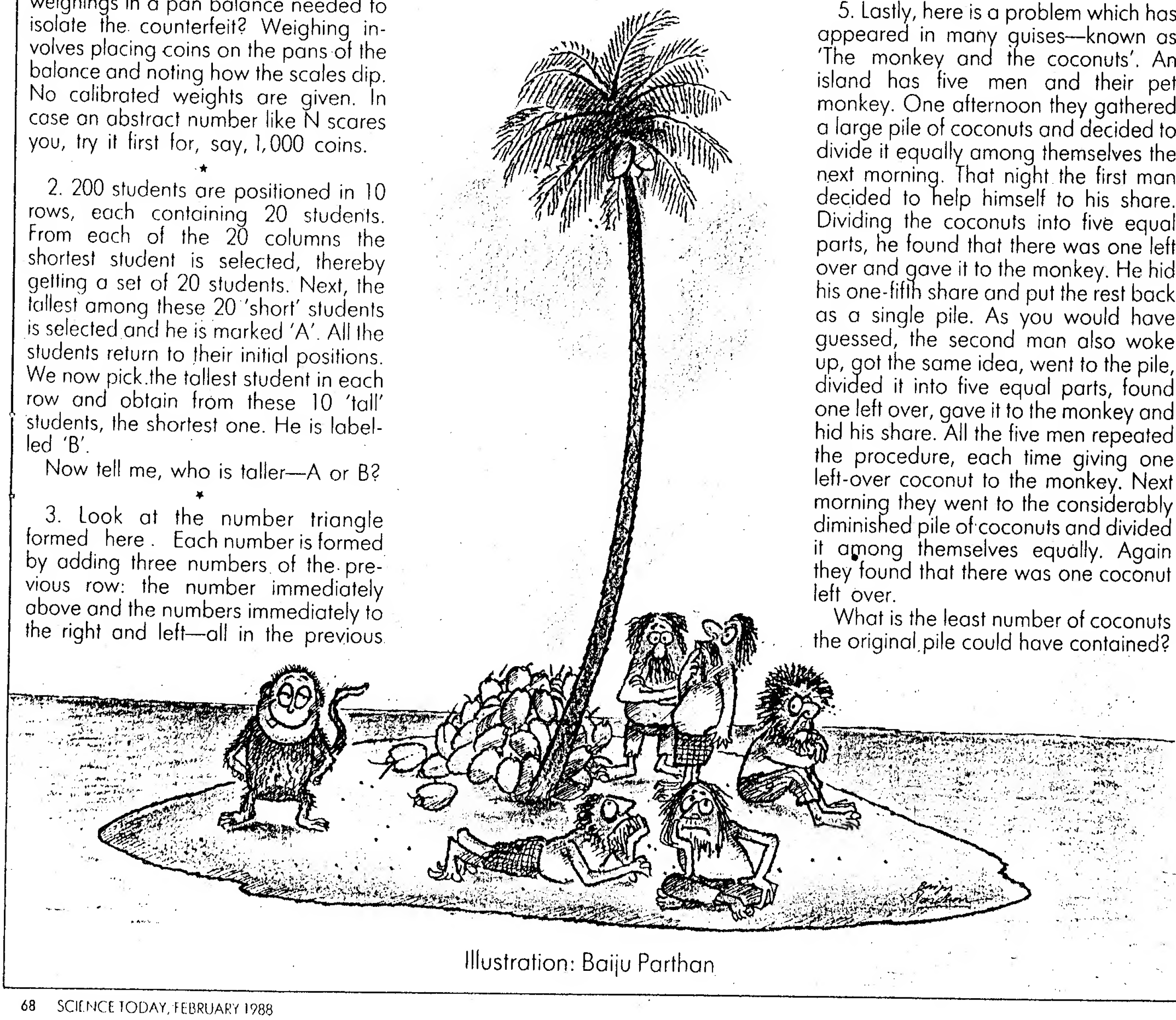


Illustration: Baiju Parthan

PLAYTHEMES

THE STRAIGHT DOPE

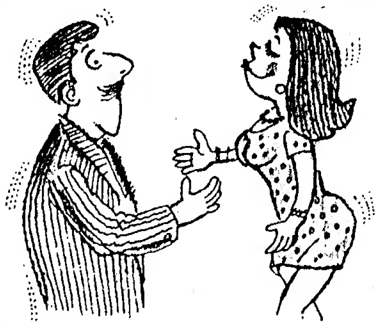
T. PADMANABHAN

Problems you can't get right for the wrong reasons

Illustrations: Baiju Parthan

THE problems below can all be solved by simple straightforward reasoning. Did I say simple? Well, not really that simple; not all of them anyway. Send the answers to SCIENCE TODAY.

1. John and his wife went to a party a few weeks back. There were four other married couples and as you would've expected, there was a lot of handshaking. No one shook hands with oneself



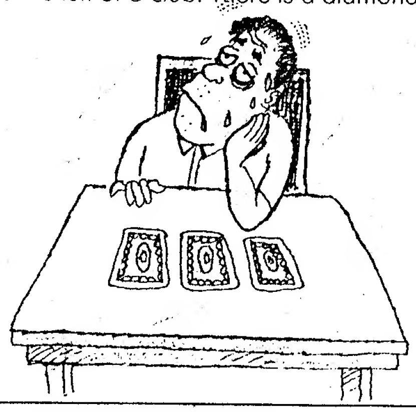
(this is just getting a bit too logical!) or with one's own spouse (reasonable, right?). No one shook hands with the same person more than once (that would be stupid, isn't it?). John, being infernally curious, asked each guest how many people he/she had shaken hands with. Surprisingly enough, each guest gave a different answer. Now tell me, how many hands did John's wife shake?

★

2. In any group of six people, there are either three mutual friends or three mutual strangers. Prove it.

★

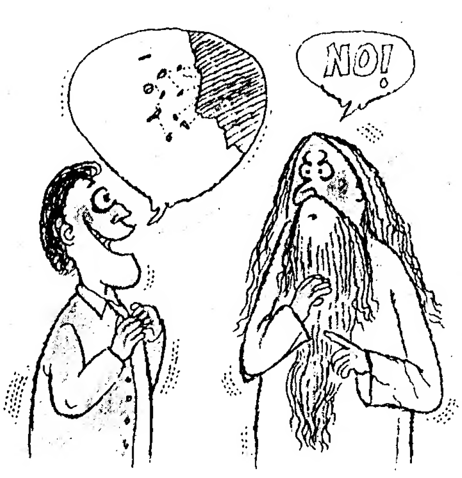
3. Three cards lie face down on a table side by side. There is a queen to the right of a king; there is a queen to the left of a queen. There is a diamond to the left of a club. There is a diamond



to the right of a diamond. Can you identify the cards?

★

4. The ancient city of Zebuka consists of 10 islands to the north of the mainland. Five of the islands had each a single bridge leading to the mainland (there are no other connections to the mainland). Four of the islands had four bridges leading from them; three of the islands had three bridges leading from them; two of the islands had two

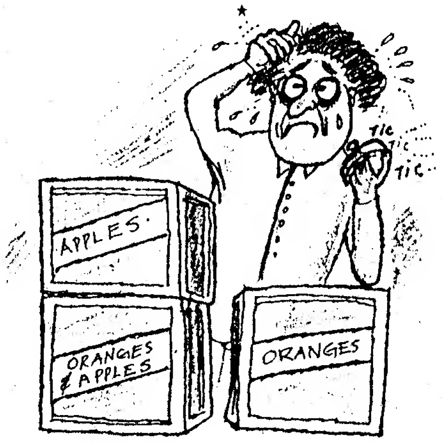


bridges leading from them and one island could be reached by only one bridge. Ancient wise men warn us not to take this description too seriously. What is wrong with the description?

★

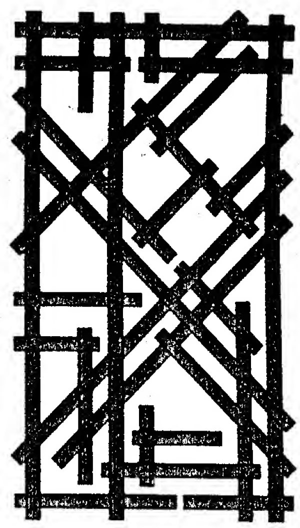
5. There is a correspondence between the symbols (A, B, C and D) and (W, X Y and Z) such that : If A is not X then C is not Y; if B is either Y or Z then A is X; If C is not W then B is Z; If D is Y then B is not X; If D is not X then B is X. What is the correspondence between these sets?

★



6. Three boxes are labelled 'Apples', 'Oranges' and 'Apples and Oranges'. Each label is incorrect. You are allowed to select one fruit from any one of the boxes. How can you label the boxes correctly? (If you take more than 10 seconds to solve this, don't bother to send in the solution!)

★



7. Figure above shows a view looking directly down upon a pile of straight rods 5 mm x 5 mm square in cross-section and of various lengths. Identify the highest point of the whole pile.

PLAYTHEMES

SCRYPT WRITER

T. PADMANABHAN

Deciphering the hieroglyphics in *The Dancing Men* enabled Sherlock Holmes to unmask the murderer of Hilton Cubitt. Code-cracking here may not yield anything quite as dramatic, but then the real joy is in the task itself, isn't it?

CRYPTAMATICS deals with cracking codes to find the hidden numbers. The clues are usually purely mathematical, as the problems below will indicate. (Yes, yes, I know I have to discuss the solutions of several previous *Playthemes*, but patience, they are coming!)

Talking of Eves

An old cryptarithm of unknown origin and of the MCP kind is as follows:

$$\frac{EVE}{DID} = .TALKTALKTALK...$$

In the above equation each letter stands for a particular digit including zero. The fraction (EVE/DID) has been reduced to its lowest terms. As the right-hand-side indicates, the decimal equivalent of this fraction has the form of a set of four digits recurring indefinitely. Your task is to reason out which letter stands for which digit.

Go ye, and multiply!

Here is a multiplication that was in vogue before the era of calculators. All the digits have been blanked out. But you are told that the only digits which appear in this crazy sum are prime numbers. Since 1 is not a prime number you are left with 2, 3, 5 or 7 to fill up the blanks. Can you reconstruct the multiplication?

```
      * * *   (×)
        * *
    -------
    * * * *
  * * * *
  ---------
  * * * * *
  ---------
```

Just A, B, C...

Here is another classical cryptamatic. Each letter stands for a particular digit. Reconstruct the multiplication.

```
A B C D E (×)
        4
---------
E D C B A
---------
```

Logical deduction

The calculation given below is perfectly correct as it stands: after all, 3 subtracted out of 11 does leave 8. However, if we treat this as a cryptamatic and let each letter represent a different digit—alas, this sum cannot be correct. Can you prove it?

```
  E L E V E N
−   T H R E E
-------------
    E I G H T
-------------
```

Provoking the ET

Every once in a while, we earthlings start wondering about extraterrestrial (ET) intelligence. Several schemes have been suggested in literature as to how one can communicate with them, as and when we find them. We certainly have to explain to the friendly ET our communication systems, cricket rules, table manners and other such important matters. On the other hand, it may be dangerous to irritate the ET with a tough coded message. What is given below is a code devised by Ivan Bell, an English teacher from Tokyo. The punctuation marks are not part of the message. The serial numbers 1 to 14 are included only for your convenience in describing your solutions. So ignore the punctuation marks and the numbers 1 to 14 while cracking this cipher. I would love to know how good are the terrestrials in deciphering such friendly messages to extraterrestrials. Find out what each line means.

Hint: This is an extremely tough problem.

```
1. A.B.C.D.E.F.G.H.I.J.K.L.N.P.Q.R.S.T.U.V.W.Y.Z.
2. AA,B; AAA,C; AAAA,D; AAAAA,E; AAAAAA,F;
   AAAAAAA,G; AAAAAAAA,H; AAAAAAAAA,I;
   AAAAAAAAAA,J.
3. A K A L B; A K A K A L C; A K A K A K A L D.
   A K A L B; B K A L C; C K A L D; D K A L E.
   B K E L G; G L E K B. F K D L J; J L F K D.
4. C M A L B; D M A L C; I M G L B.
5. C K N L C; H K N L H. D M D L N; E M E L N.
6. J L A N; J K A L AA; J K B L AB; AA K A L AB.
   J K J L BN; J K J K J L CN. FN K G L FG.
7. B P C L F; E P B L J; F P J L FN.
8. F Q B L C; J Q B L E; FN Q F L J.
9. C R B L I; B R E L CB.
10. J P J L J R B L S L A NN;
    J P J P J L J R C L T L A NNN.
    J P S L T; J P T L J R D.
11. A Q J L U; U Q J L A Q S L V.
12. U L WA; U P B L WB; AWD M A L WD L D P U.
    V L WNA; V P C L WNC. V Q J L WNNA;
    V Q S L WNNNA. J P EWFGH;
    S P EWFGH L EFGWH.
13. GIWIH Y HN; T K C Y T. Z Y CWADAF.
14. D P Z P WNNIB R C Q C.
```

PLAYTHEMES

BACKLOG

T. PADMANABHAN

AT last here come the answers to the June '87 problems. A bit late, aren't we? Never mind, we'll try to catch up with the backlog. So then, on to the answers.

1. This puzzle can be solved in 15 moves. The answer is most easily stated by indicating the number of the empty cell after each move. In this notation the answer is 4,3,5,6,4,2,1,3,5,7,6,4,2,3,5,4.

The numbers refer to Fig.1 in the June '87 issue. To begin with, 4 was vacant; we move the red counter at 3 to 4 vacating 3 and so on. Correct answers were sent by Rajesh Jaluka (Calcutta), Pushpangadan (Alleppy), R. J. Kadam (Pune), D. Janah (Calcutta), K. P. Namrata (Nagpur), R. Parekh (Calcutta), V. M. Gadre (Delhi), M. L. Munvar (Bombay), R. P. Nirgudkar (Pune), and V. G. Annigeri (Bagalkot).

2. As I said, this problem is ridiculously simple if you have solved the previous one. The trick is to use the solution of the previous problem. Consider the initial position of Fig. 2 in the June '87 issue. The red and green counters of the 4th row can be interchanged just as in the previous problem without disturbing anything else. Now think of column d. Again, the red and green counters in this column can be interchanged by our solution to problem 1. But as we do this, various cells in column d will fall vacant. Whenever one cell falls vacant in column d, the counters in the corresponding row can be interchanged. Thus, for example, once we move d3 to d4, d3 will be vacant and the red and green counters of the third horizontal row can be interchanged. We need 15 moves for each row and thus a total of 7×15=105 moves for all the rows. In addition, we need 15 moves for the vertical column—thus totalling 120 moves. The correct strategy was adopted by R. Jaluka, R. J. Kadam, D. Janah, Ranjan Parekh, M. L. Munvar, R. P. Nirgudkar, V. G. Annigeri, and V. M. Gadre. Interestingly enough, not everyone who has solved the first problem has solved the second!

3. The 46-move solution is given below. Each move is indicated by giving the letter of the cell from which the counter is moved.

H h g I F f c I C B H h I G D F f e h b a g I
G A B H E F f d g I H h b c I C F f I G H h

If you leave out 'I' which is the central square, then you can see a remarkable symmetry in the above solution. The second line is just the first one in reverse replacing capital alphabets by lower case ones; 'I', of course is the same as 'i' since it is the central square! Correct answers were given by V. G. Annigeri, M. L. Munvar, D. Janah and R. J. Kadam.

4. The classic solution to the Solitaire problem is the following: 19-17, 16-18, (29-17, 17-19), 30-18 27-25, (22-24, 24-26), 31-23, (4-16, 16-28), 7-9, 10-8, 12-10, 3-11, 18-6, (1-3, 3-11), (13-27, 27-25), (21-7, 7-9), (33-31, 31-23), (10-8, 8-22, 22-24, 24-26, 26-12, 12-10), 5-17. It is usual to treat a connected sequence of 'jumps'—indicated by brackets in the above solution—as a single move. With this convention the above solution has 19 moves. Giving this solution in the magazine *Strand* (April 1908), H.E. Dudney added: "I do not think the number of moves can be reduced." Well, four years later, E. Bergholt published an 18-move solution! Probably the readers would like to have a go at beating Dudney. Incidentally, it can be proved that 18 moves is the minimum. Correct solutions were from V. G. Annigeri, M. L. Munvar, R. P. Nirgudkar, M. S. Sabane (Bombay), Ranjan Parekh, K. P. Namrata and D. Janah.

Complete solutions to all problems were sent in by V. G. Annigeri, M. L. Munvar and D. Janah. Congrats!

All the above problems are classics. The first three (and similar ones) are discussed, for example, by Rouse Ball in his *Mathematical recreations and essays*. Several books discuss Solitaire, one of the earliest being *The game of Solitaire* by E. Bergholt, New York: Dutton, 1921.

Doublets and other projections

Now for the answers to the posers of the August 1987 issue.

1. The trick here is to use the fact that projection by parallel rays leaves the ratio of areas unchanged. Consider a right-angled triangle with the maximum-area-ellipse inscribed in it, situated in plane A (Fig. 1). Project

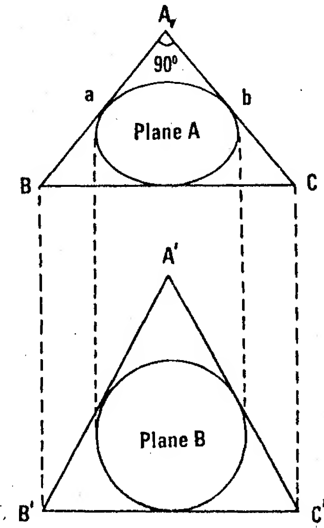


Fig.1

this figure into plane B such that the ellipse becomes a circle. We now show that the triangle becomes equilateral. If A is the area of the projected triangle, P its perimeter and R the radius of the inscribed circle, then we are given that $\pi R^2/A$ is maximum. Since $A = \frac{1}{2}RP$, R/P must be maximum and therefore PR/P^2 or equivalently, A/P^2, is maximum. Since triangles with the maximum dimensionless ratio A/P^2 are equilateral, we know that our triangle must become equilateral on projection. The original ratio of areas is same as the ratio of areas of the circle and the equilateral triangle. Simple geometry gives $\pi R^2/A = \pi/3\sqrt{3} \simeq 0.605$. Thus the area of the inscribed ellipse is $0.605 \times ab/2 \simeq 0.30ab$. Unfortunately, nobody got this answer right.

2. If the numbers were $10a+b$ and $10b+a$ their product will be $10a^2+101ab+10b^2$. Since this is not of the form $a^2+2ab+b^2$ several readers concluded that it is 'obviously not a

perfect square'. Such reasoning, however, is faulty. To see this, consider the product of two 3-digit numbers, $100a+10b+c$ and $100c+10b+a$ with the product $100(a^2+b^2+c^2)+1010(ab+bc)+10001ac$. This is certainly not in the form $a^2+b^2+c^2+2(ab+bc+ac)$ and hence (by the previous argument) it cannot 'obviously be a perfect square'. But alas, for $a=1$, $b=6$ and $c=9$, the numbers 169 and 961 multiplied to give the perfect square $(403)^2$!

Now that you know the hidden trap in the problem, why not try again? I will discuss the answer in a later issue.

3. The word square was solved by several people. The correct set of words from top to bottom are: LASHES, ARTERY, STORMS, HERMIT, ERMINE, SYSTEM. Correct answers were from T.R. Unni (Bombay), O. Rajopadhyaya (Nepal), N. Venkatesan (Annamalainagar), Avik Ghosh (UP), Ratnaprabhu (Ahmedabad), Rajnarayanan (Shornur), J. Nandi (Calcutta), Verghese George (Mysore), Anshu (Chanda), and R. Jaluka (Calcutta).

4. The only way to get 100 is with eight hits. Score 13 six times and 11 twice. This problem was solved by B.H. Bagawan (Samalkota), Rajopadhyaya, Venkatesan, Avik Ghosh, Ratnaprabhu, Ulhas Bhatt (New Delhi), Rajnarayanan, J. Nandi, Verghese George, Anshu, R. Jaluka and Ilamuguru (Trichy).

5. The *simplest* solution to this problem is a solid cylinder with a flat slice taken out from one side (Fig.2). Correct or almost correct answers were given by Verghese George, J. Nandi, Ratnaprabhu, Avik Ghosh (UP) Venkatesan, Rajopadhyaya and T.R. Unni.

6. A general seven-piece pattern of cutting any obtuse-angled triangle is shown in Fig.3. (In fact 7 is minimal.) Clearly, we need a line dividing the obtuse angle. But if we let the line go all the way to the other side, we will create another obtuse-angled triangle (in general) and thus the solution won't be minimal. The trick is to terminate the line inside the triangle. This solution is due to Wallace Man-

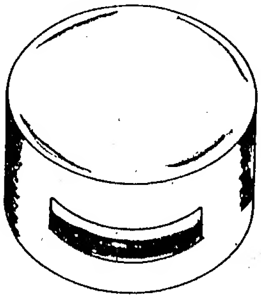


Fig. 2

heimer (*American Mathematical monthly*, November 1960, p. 923). Several readers sent me 'proofs' that this is impossible, the most common error being the assumption that the line dividing the obtuse angle must meet the opposite side. No one completely solved the problem though J. Nandi came pretty close.

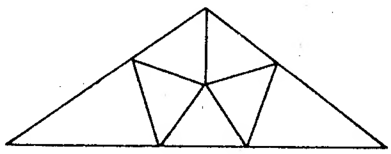


Fig. 3

7. Martin Gardner discussed Lewis Carroll's answer to doublets in his column in *Scientific American*. Soon afterwards readers sent in shorter solutions. The best ones for this problem were the following:

(i) GRASS-CRASS-CRESS-TRESS-TREES- TREED- GREED- GREEN. No one could improve on this. The best I got are in eight steps: (a) GRASS-CRASS- CRESS-TRESS-TREES-FREES-FREED-GREED-GREEN (Ashish Mahabal, Nagpur). b) GRASS-GROSS-GROWS-GROWN-GROAN-GROAT(an old doubloon)-GREAT-GREET-GREEN. (Avik Ghosh). Anshu could produce dry Indian grass in seven steps: GRASS-GLASS-GLOSS-GROSS-CROSS - CROWS -CROWN-BROWN. Question: can Anshu, or anybody else, do it in less steps?

(ii) Several readers equalized with the five-step solution of Martin Gardner: APE-APT-OPT-OAT-MAT-MAN. They are Unni, Ratnaprabhu, A. Mahabal and Avik Ghosh.

(iii) Gardner has the following six-stepper involving OYE (Scottish word for grandchild); ONE-OYE-DYE-DOE-TOE-TOO-TWO. The best attempt was Mahabal's: ONE-OWE-EWE-EYE-DYE-DOE-TOE-TOO-TWO.

(iv) BLUE turns into PINK in seven steps: BLUE-GLUE-GLUT-GOUT-POUT-PONT-PINT-PINK. The best reply I received was BLUE-BLUR-SLUR-SOUR-SOAR-SOAK-SOCK-SICK-SINK-PINK from Ratnaprabhu, Rajopadhyaya, Rajnarayanan.

(v) WINTER can be changed to SUMMER with "remarkably well known" words in just eight steps. WINTER-WINDER-WANDER-WARDER-HARDER-HARMER-HAMMER-HUMMER-SUMMER. The best I got (from Rajopadhyaya) is similar to the above with the change: HARDER-HARPER-HAMPER-HAMMER.

The first problem was discussed by Graham in the magazine *Dial*; questions 4 and 5 are adapted from the puzzles originally devised for Mensa-Superbrain contests.

BRAIN TEASER

DEBABRATA CHATTERJEE

There was general dissatisfaction after the election of the women's club of our town. The executive body was dominated by the relatives of one member of the executive committee, so the ordinary members felt.

The five posts that constitute the executive committee are: President, Secretary, Treasurer, Magazine Editor and Social Secretary.

The following additional data are also known:

1. The President declared, "But I have no relative in the committee. None of the four other executive members, Mrs Gupta, Mrs. Singh, Mrs Thakur and Mrs. Sharma is my relative.

2. One of the members protested and said, "But Mrs. Gupta is the daughter of the editor, and Mrs. Sharma is the sister-in-law of the editor."

3. Another ordinary member remarked, "The secretary is the aunt of the treasurer."

4. "That is a wrong statement," protested the Social Secretary. And then added, "The statement of Miss Hazarika is equally applicable to me as well. I also do not have any relative in the committee.."

5. "We sympathize with you for that, Mrs. Thakur," said an ordinary member sarcastically.

Can you now find out who holds which post?

PLAYTHEMES

NUMBER GAME'S UP

T. PADMANABHAN

BEFORE going on to discuss the answers to the questions posed in the September '87 issue, we first set forth a few easy puzzles for you to crack.

(i) Is there a way of slicing a cube by a plane such that the cross-section is a regular hexagon? While at it, can you slice a torus such that the cross-section represents two circles which intersect?

(ii) You are given a triangle ABC. Find a point P in the same plane such that PA + PB + PC has the minimum possible value.

(iii) Drunkards are notorious for walking randomly. Two drunkards start from a particular point. One guy takes 70 random steps each of unit length while the other guy takes 30. (A random step is one which could be in any direction in the plane, with respect to the previous step; but all steps are supposed to have the same length.) What is the expected distance between our drunken friends?

The September '87 *Playthemes* on number games produced an enormous response. Let us now run through the answers.

1. Two different ways of expressing 19 with four 4's are; $19=4!-4-4\,4$; $19 = (4+4-\cdot4)/\cdot4$. Clearly the second solution can be repeated with any digit other than 4. One way of expressing 64 with two 4's is

$$64=\sqrt{\sqrt{\sqrt{4^4}}}$$

One interesting alternative is to use the double factorial symbol: $64=4!! \times 4!!$ (double factorial implies product of previous integers which are all even/odd; $5!!=5\times3\times1$, $6!!=6\times4\times2\times1$ etc). There are, of course, several other solutions.

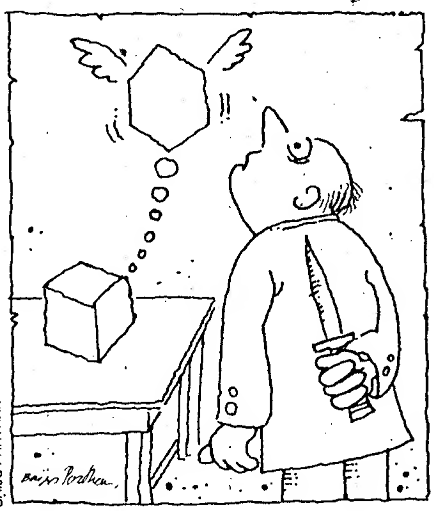
More than 40 readers answered this correctly! So let me just discuss interesting alternatives. The most imaginative answer for expressing 64 with two 4's came from N.M. Dongre, Bombay. He says it is just '44' if you use base 15! Another bit of lateral thinking (but previously known) was to write ♭4 as our solution (A.K. Gupta, Calcutta, B.J. Deshmukh, Pune). Several readers solved the above problems using the fact that infinite number of repeated square roots reduce any number to unity (Sujata Bhagwat, Jabalpur, Mukesh Kumar, Delhi). One example of this technique is:

$$19=\frac{4+4}{\cdot4}-\sqrt{\ldots\sqrt{\sqrt{4}}}$$

2. The four- and three-digit numbers with the required property are: 7641 and 954 ($7641 - 1467 = 6174$; $954 - 459 = 495$). While several readers sent correct answers, M. Kalyanaraman, Madras and S.V.N. Murali, Bangalore, deserve special mention for detailed analysis.

3. The solutions with minimum number of symbols are the following: $123 - 45 - 67 + 89 = 100$; $98 - 76 + 54 + 3 + 21 = 100$. Nearly 25 readers sent in right answers, among whom M. Kalyanaraman gave a very methodical analysis of the problem.

4. A perfect square cannot have all its digits identical. To see this, note that the last two digits of a square are decided by the last two digits of its square root. Among the squares of all two-digit numbers only 4 is repeated in the end. Thus a perfect square with all its digits identical—if it exists—must have the form 44... 4. But 44... 4 = 4 × 11... 1. Since 4 is a perfect square, this implies that 11... 1 should also be a perfect square. This is impossible because we have just now reasoned that perfect squares with identical digits must have all digits equal to 4.



BAIJU PARTHAN

A perfect square cannot have its last two digits both odd (and hence it cannot have all its digits odd). One way to prove this is as follows: Since the last digit is odd, the perfect square must be the square of an odd number. Let us write this number as $10x + q$ where q is an odd *digit* while x is any *number*. Its square is $100x^2 + 20xq + q^2$. Let us examine the last two digits of this number. $100x^2$ does not contribute to the last two digits. Writing $20xq$ as $10\times2xq$ we see that the unit place digit is zero and the ten's place is even. Similarly q^2 has either no ten's place digit (for $q=1, 3$) or it contributes an even number to the ten's place (for $q=5, 7, 9$). Either way the ten's place of $(10x+q)^2$ has an even digit when q is odd. This proves the claim.

Several variants of the above proofs have come from readers. The crispest analyses were from N.M. Dongre, R.V. Pradhan (Madras), T.N. Ramprasad (Madras), S. Gupta (Rajkot), Mukesh Kumar and Tushar Samant (Bombay).

5. It is easy to see that four markings are needed. With n markings (not counting the end points 0 and 12) we can get $1/2\,(n+2)\,(n+1)$ pairs. Thus with three markings, we can get only 10 different lengths. With four markings there are several solutions. For example, the markings can be at 1, 4, 7 and 10. Given any set (a_1, a_2, a_3, a_4), we can construct a new solution replacing a_i by $(12-a_i)$. More than 20 readers sent in correct answers.

The following readers solved all the problems correctly: Kalyanaraman, Santosh Iyer (Jaipur), Jaideep Nandi (Calcutta), Dongre, Mukesh Kumar and S. Gupta. Congrats! The problems discussed here are all classics and have appeared in literature several times: especially in Dudney's problem collections and in Martin Gardner's column featuring Dr. Matrix.

PLAYTHEMES

OUTSIDE IN

T. PADMANABHAN

CONTINUING the spree of clearing up our backlog, here come the answers to the December '87 column.

1. The right way to tackle this problem is to think how many free edges and sides the cutting will produce; this is easier to figure out than the final shape. The answer is a flat sheet of paper with a hole in it. That is, a shape with two sides, two edges and no twists. The four corners formed at the intersection of dotted lines will be the four corners of the square. This surprising result arises because loop A can be opened out and untwisted the moment B is slit along the dotted line (see Fig.1 of the Dec. '87 issue).

2. We get two loops, unlinked, one with a right-handed twist and the other with a left-handed one. Here too the trick is to study what happens to the edges. This is shown in Fig. 1. We see that the edges form two unlinked loops with twists in them. See Fig. 1 and Fig. 2.

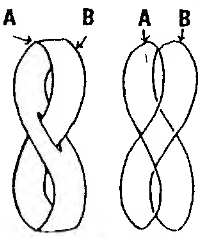


Fig. 1a *Fig. 1b*

Right **Left**

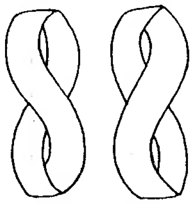


Fig. 2a *Fig. 2b*

3. This problem is really a sitter if you think about it properly. We have to fold the cloth belt in such a way that it has a rectangular shape and uniform thickness. The simplest way to do it is shown in Fig. 3a and 3b. The

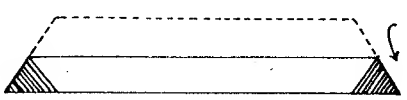


Fig. 3a

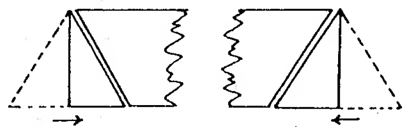


Fig. 3b

first fold along the length doubles the thickness everywhere except at the shaded edges. The second fold doubles up the thickness at the edges and produces the rectangular shape. Now one can roll the belt in peace.

4. The torus turnover does not delink the circles. When you 'invert' the torus through a hole the two circles drawn on it merely exchange places but remain linked. This is perfectly obvious if you think of the torus as a rectangular sheet of paper with opposite edges identified.

5. The best method of attack is again to think of a torus as made from a rectangular piece of paper by pasting the opposite edges together (we first paste the lengths together getting a cylinder and then make a 'cycle tube' out of this cylinder). Inverting the torus interchanges the length and width of the original sheet. Thus, if we started out with a square sheet the proportion will remain unchanged on inversion. It is also necessary that the hole runs one full 'half-width'.

6. Passing the key through the loop involves the steps shown in Fig. 4a, 4b, and 4c. We first pass the key through the loop and bring it to position A. Now comes the crucial manoeuvre. Pull the double cord at A, B and thus bring the loop through and out of the keyhole (see Fig. 4b).

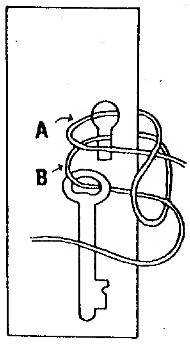


Fig. 4a

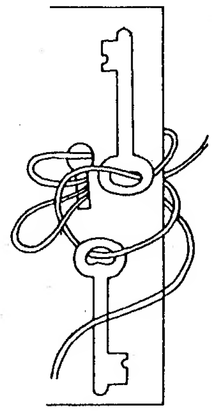


Fig. 4b

Now we can move the key up the cord through both the loops to the position shown in the figure. We now reverse the manoeuvre by grasping the cords on the other side of the door and pulling the loop back through the hole. This restores the cord to the original position. But now we can slide the key along as shown in Fig. 4c!

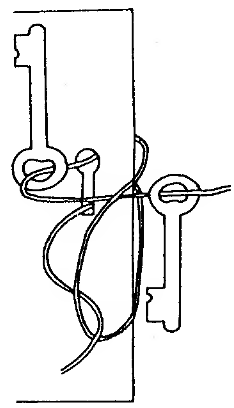


Fig. 4c

This last problem is a creation of Martin Gardner and has appeared in *Scientific American*. The other problems are due to Stephan Barr, the American puzzlist.

The *best—almost complete—* answers were received from N. S. Kar (Kalyani University) and Shafeek Mohammad (Tellicherry). Congrats!

PLAYTHEMES

DOUBLESPOKE

T. PADMANABHAN

THE October *Playthemes* received an all-time low of just two responses! What has happened to all of you out there? I refuse to believe the questions were too tough. Anyway, here are the answers.

1. The simplest way to attack this problem is to denote regions of a map by points. Whenever two regions are adjacent in a map we will connect the corresponding points by a line. Thus any given map with n regions can be converted into a figure containing n points with interconnecting lines. Of course, the lines are not supposed to cross except on the n points. To construct a map of five regions such that each region is adjacent to the other four is now equivalent to the following problem: Mark five points on a plane and connect each one to the other four by non-intersecting lines. The impossibility of this task can be seen as follows:

Consider any three points which are interconnected. This should look like Fig. 1. We now add a fourth point D and connect it to A, B, and C. This can be done in two equivalent ways as shown in Fig.2 (a) and 2 (b). But note that, in either approach, one point gets encircled by the lines. It is now impossible to add a fifth point and connect it to A, B, C and D. For example, if we put it outside the configuration it can't be connected to D in Fig. 2(a) or to C in Fig.2(b).

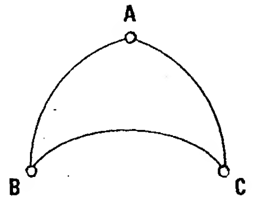


Fig. 1

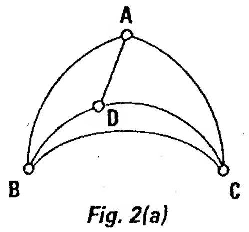


Fig. 2(a)

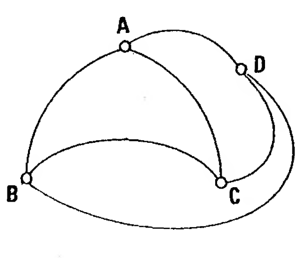


Fig. 2(b)

I leave it to you to figure out why this doesn't prove the four colour theorem.

2. The result—that two colours are enough to colour maps made of intersecting straight lines—can be easily proved using the principle of mathematical induction. Suppose you have some map with k straight lines which could be coloured with just two colours. Let us add one more straight line anywhere in the map (see the thick black line in Fig 3). This line divides the original map into two maps. Each of these maps, taken individually are properly coloured with two colours. All we have to worry about are the regions neighbouring the straight line we have drawn. As it is there are regions around this line which violate our condition that no two adjacent regions should have the same colour. But this is easily taken care of by just interchanging the two colours on any of the two maps. This is shown in Fig.4. We have thus proved that given a map of k lines which is properly coloured one can properly colour a map of (k+1) lines. It is trivially true that a map with one line can be coloured with two colours. Thus it follows that any map made of intersecting straight lines need only two colours.

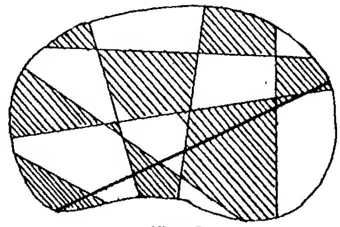


Fig. 3

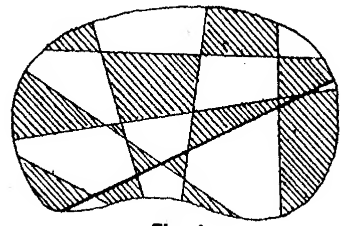


Fig. 4

3. The lines drawn on Fig. 5 tell the tale. Since the areas of regions marked 1, 2, 3 are equal to those of the regions marked 4, 5, 6, the answer is obvious.

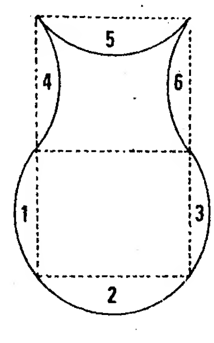


Fig. 5

SCRABBLING

HERE is a puzzle for all scrabble players: what is the highest score a player can obtain in a single turn of play in scrabble?

You are expected to give (i) the position on the board before the player puts down his word, (ii) the letters he holds and, of course, (iii) the word he makes. Make sure that all the words are 'legal' and that the position uses only the standard set of SCRABBLE chips, for instance, it can have at the most nine A's, two B's etc.

Most of the known high-score-solutions to this question use words which are quite uncommon. The idea is to find combinations which use only 'common' words but still produce a 'high enough' score. That makes this question fairly open-ended. I will run all the interesting answers I receive. So, have a go!

T.P.

PLAYTHEMES

CLASSICAL ORIGINS

T. PADMANABHAN

YES, fresher pastures it will be. Let us take a break from abstract mathematics and see whether we can solve problems in an everyday setting. Some of these are classics and discussed extensively in several places. But never mind. There is always scope for some originality.

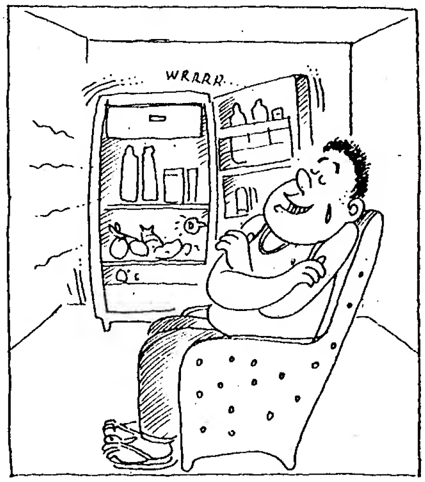
1. You drink coffee from your favourite cup which has a mark just below the rim. You always fill the cup up to this mark and slowly consume

the poison, thinking deep thoughts. Suppose you are doing the same thing in a colony set up on the surface of the Moon. Will you be drinking more coffee, less coffee, or will it be the same?

2. You are in charge of providing colour stage lights for a local dance festival in the theatre. The pretty things which are going to dance insist the stage being bathed in green light,

but alas, you have no green filter. But you do have a blue and a yellow filter. "Thank God," you reason, "all I have to do is to put the blue filter in one lamp and yellow in the other and direct both the lamps on to the stage. The blue and yellow will mix to give a perfect green." Will this idea work? Remember that you can always put both the filters in front of the same lamp and get the green. If you think the first idea won't work then you have to explain the precise difference between the two.

3. Here is a classic which leads to heated arguments whenever people start on it. On a hot summer day your room is at 35°C. You plan to cool it by the following ingenious method. Close all windows and doors, put your refrigerator in the middle of the room, keep the refrigerator door open and switch it on. (We will assume that



the room is thermally insulated but the windows and doors have small holes to let air in and out; after all, you have to breathe!) When things have reached an equilibrium, assuming they do, what will be the temperature of the room? Will it be cooler? Hotter? Or the same?

4. An object weighs less in Madras than in Bombay because of at least two different reasons. Find them.

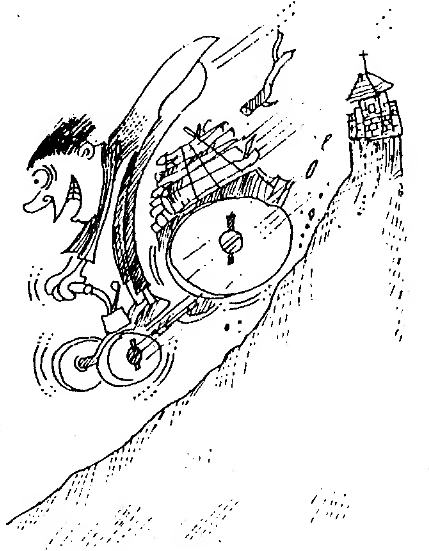
5. A mass of interstellar cloud

Illustrations: Baiju Parthan

made of small black soot like dust approaches the Earth. On calculating its trajectory, scientists realize that it is going to settle as a thin layer over the North Polar region. 87/88

Describe what will happen to the Earth's weather. Will the consequences be very different if the cloud settled around the South Polar region?

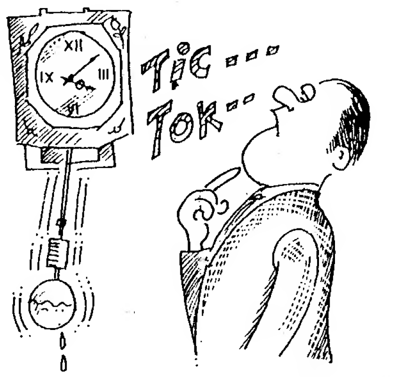
6. My friend lives on the top of a mountain near a remote village. He chops wood at the mountain top and sells it to the villagers to make his living. He astonished me by describing his *modus operandi*: "You see, I have these carts with a dynamo and chargeable battery. I put all the wood in the cart and roll down the slopes of the mountain with it. The rotating wheels are coupled to the dynamo which charges the batteries. I go around the village, selling the wood and in the evening use the battery power to drive the cart up." I thought this was funny. It should take the same amount of energy in the downhill and uphill path so he couldn't be



roaming around in the village selling wood and still expect to go back home. "But," he explained, "the cart weighs less in the evening. After all, I sell a lot of wood."

Is my friend pulling my leg or will this scheme really work?

7. The good old grandfather clock depends on the length of the pendulum for keeping its time. Let us tinker



with it. At the end of the pendulum we attach a small hollow sphere filled with water. Then we make a hole in the sphere so that the water can slowly leak out drop by drop. Now tell me, what happens to the working of the clock as the water leaks out? Does it go slower, faster or remain same?

8. Margaret was educating Dennis on the intricacies of an electric motor.

"It's quite simple. A current flows through this coil which is surrounded by magnets. So the coil begins to rotate."

"What would happen if I put in a stronger magnet?"

"The motor would run faster, you dumb."

"Really? Where does the extra energy come from?"

9. We conclude with an old classic. Can you propel a sail-boat by a battery-operated fan kept on the boat? Before you start on action-reaction, energy conservation and all the rest of it, remember that the battery is running down and the boat is not isolated from the atmosphere.

FUN WITH MATHS

SQUARE PAIR

P. K. MUKHERJEE

IF 2n be the total number of digits in any of the two numbers in a pair where X and Z are the first n-digit blocks of the respective numbers and Y the common n-digit block they end in, then these numbers XY and ZY can be expressed in the following manner

$XY = X^2 + Y^2 = Y + 10^nX$

$ZY = Z^2 + Y^2 = Y + 10^nZ$

Subtracting the second of these equations from the first, we have

$X^2 - Z^2 = 10^n\ (X-Z)$ or,

$X+Z = 10^n$ ...(A)

If from the same equations 10^n be now replaced by X+Z, we obtain

$XZ = Y\ (Y-1)$ ...(B)

The same two equations will give for X and Z the values

$$X = \frac{10^n - \sqrt{10^{2n} - 4Y(Y-1)}}{2}$$

$$Z = \frac{10^n + \sqrt{10^{2n} - 4Y(Y-1)}}{2}$$

For n=2, it can easily be verified that the only value of Y that makes the radical a perfect square is Y=33 which yields for X and Z the values 12 and 88 respectively. Hence, the number pair corresponding to n=2 is (1233, 8833). For n=3, only Y = 100 makes the radical a perfect square for which the values of X and Z work out to be 10 and 990 respectively. Therefore, the number pair corresponding to n=3 is (10100, 990100) For n=4 the only number pair possible is (5882353, 94122353). For n=5, however, three number pairs can exist, namely, (99009901, 9901009901); (17650381 25, 8235038125) and (2584043776, 7416043776).

Other number pairs can similarly be obtained for higher values of n. However, complications increase as the order of n becomes large and, therefore, one may have to take the help of computers.

Consider some of the properties of such a number pair (XY, ZY).

If we multiply the numbers XY and ZY, we have

$$\begin{aligned}(XY)\ (ZY) &= (Y + 10^nX)\ (Y + 10^n\ Z)\\ &= Y^2 + 10^nY\ (X+Z) + 10^{2n}\ XZ\\ &= Y^2 + 10^n\ Y\ 10^n + 10^{2n}\ Y\ (Y-1)\\ &= Y^2 + 10^{2n}\ Y^2\end{aligned}$$

As Y has n digits, Y^2 can have either 2n or (2n-1) digits. If Y^2 has 2n digits then the product (XY) (ZY) will be of the form Y^2Y^2; if it has (2n-1) digits the product will have the form Y^20Y^2.

Interchanging the n-digit blocks in the respective numbers, the resulting numbers become YX and YZ respectively. Multiplying them leads to

$$\begin{aligned}(YX)\ (YZ) &= (X + 10^nY)\ (Z + 10^nY)\\ &= XZ + Y\ (X+Z)10^n + 10^{2n}\ Y^2\\ &= Y\ (Y-1) + Y\ 10^{2n} + 10^{2n}Y^2\\ &= Y\ (Y-1) + 10^{2n}\ Y\ (Y+1)\\ &= Y_1 + 10^{2n}\ Y_2 \text{ where}\end{aligned}$$

$Y_1 = Y\ (Y-1) = Y^2-Y$

$Y_2 = Y\ (Y+1) = Y^2+Y$

When Y^2 has 2n digits, Y_1 also possesses 2n digits. The product (YX) (YZ) in that case is of the form Y_2Y_1, whereas in the event of Y^2 possessing (2n−1) digits the product will have the form Y_20Y_1.

Interchanging the n-digit blocks and squaring generates the numbers $(YX)^2$ and $(YZ)^2$ respectively. If to these numbers are added the squares of the unchanged numbers the resulting numbers become $(YX)^2 + (ZY)^2$ and $(YZ)^2 + (XY)^2$ respectively. We can then have

$$\begin{aligned}(YX)^2 + (ZY)^2 &= (X+10^nY)^2 + (Y+10^nZ)^2\\ &= X^2 + 10^{2n}\ Y^2 + 2(X\ Y)\ 10^n + Y^2 + 10^{2n}\ Z^2 + 2(Y\ Z)10^n\\ &= X^2 + Y^2 + 10^{2n}\ (Z^2+Y^2) + 2Y\ (X+Z)10^n\\ &= XY + 10^{2n}\ (ZY+2Y)\end{aligned}$$

Similarly,

$(YZ)^2+(XY)^2 = ZY + 10^{2n}\ (XY + 2Y)$

It is clear that the interchanged numbers get corrected. Also, by subtracting 2Y from the first n-digit blocks of the results the two unchanged numbers ZY and XY respectively are recovered.

The numbers XY and ZY can be 'weighted' by $X\ 10^{-n}$ and $Z\ 10^{-n}$ in two possible ways. The weighted additions are

$X\ 10^{-n}\ (XY) + Z\ 10^{-n}\ (ZY) = Y + X^2+Z^2$

$Z\ 10^{-n}\ (XY) + X\ 10^{-n}\ (ZY) = 2Y^2-Y$

By adding the squares of the two numbers we have

$$\begin{aligned}(XY)^2 + (ZY)^2 &= (Y + 10^n\ X)^2 + (Y + 10^nZ)^2\\ &= Y^2 + 10^{2n}X^2 + 2\ (Y\ X)\ 10^n + Y^2 + 10^{2n}\ Z^2 + 2\ (Y\ Z)10^n\\ &= 2Y^2 + 10^{2n}\ (X^2+Z^2) + 2Y\ (X+Z)\ 10^n\\ &= 2Y^2 + 10^{2n}\ (X^2 + Z^2 + 2Y)\end{aligned}$$

PLAYTHEMES

OLYMPIAN ANSWERS

T. PADMANABHAN

THE February instalment evoked considerable response from readers. On, then, to the answers.

1. If the number of coins N is less than or equal to 1/2 (3^n-3) then the counterfeit can be detected (and determined to be heavy or light) by n trials. Otherwise n trials will not, in general, be sufficient. In the case of 1000 coins, seven weighings will be needed. The proof of this result, however, is too long to be given here. I will merely suggest the line of attack. The basic strategy is to solve the problem for k coins when you are also provided with k' coins which are known to be genuine. Using the result for this case and choosing the initial weighing with x coins in each pan where $2x \leq 3^{n-1}$ and $N-2x \leq 1/2\,(3^{n-1}-1)$, the task can be achieved.

Correct answers were given by Santosh Iyer, Jaipur, and M. Kalyanaraman, Madras. Ram Prasad, Madras, and Sujata Bhagwat, Jabalpur, solved it almost correctly.

2. B must be taller than A. If A and B stand in the same row then B is taller than A since B is the tallest in the row. If A and B are in the same column then again B is taller because A is the shortest in the column. Finally, if A and B are not in the same column or row let C be the student standing in the same column as A and same row as B. Then, B (tallest in the row) is taller than C and C is taller than A (shortest in the column) making B taller than A. Thus in every case B is taller than A.

Correct answers were sent by T.N. Mukund, Bombay, Santosh Iyer, Ravindra N., Visakhapatnam, Ram Prasad, Janakiraman, Madras, and M. Kalyanaraman.

3. There are several ways of solving this problem. The simplest probably is the following. Beginning from the third row of the triangle (see fig.)

$$
\begin{array}{c}
1\\
1\;1\;1\\
1\;2\;3\;2\;1\\
1\;3\;6\;7\;6\;3\;1\\
\cdots\cdots\cdots
\end{array}
$$

write out the first four numbers of each row using E for an even number and O for an odd number. Thus, in the third row 1 2 3 2 becomes OEOE etc. It is trivial to see that the fifth row again has OEOE and that all the five rows have at least one E. The pattern merely repeats itself after this. Therefore, every row starting from the third will have an entry E, that is, an even number. In fact, we can make an even stronger statement: Every row beginning with the third will contain an even number within the first four entries in that row.

T.N. Mukund, Santosh Iyer, Ravindra N., and Ram Prasad answered this question correctly.

4. Suppose that the integer N is divisible by all numbers $m \leq \sqrt{N}$; let there be l prime numbers less than $\sqrt{N}$ and let k be the least common multiple of all the divisors. Then it is easy to see that $(\sqrt{N})^l < k^2$. But k is less than or equal to N. So we must have $(\sqrt{N})^l < N^2$ implying $l < 4$. Since $P_1, P_2, \ldots P_l$ are the primes less than $\sqrt{N}$ the fourth prime, $P_4=7$ (corresponding to $l=4$) must satisfy: $7>\sqrt{N}$. In other words, our number N must be less than 49. It is now a simple matter to verify that the following integers satisfy the requirement: 24, 12, 8, 6, 4, 3, 2.

Correct answers were received from Ram Prasad, Kalyanaraman, Sujata Bhagwat, A.K. Sahu, Calcutta, Ravindra N. and Santosh Iyer.

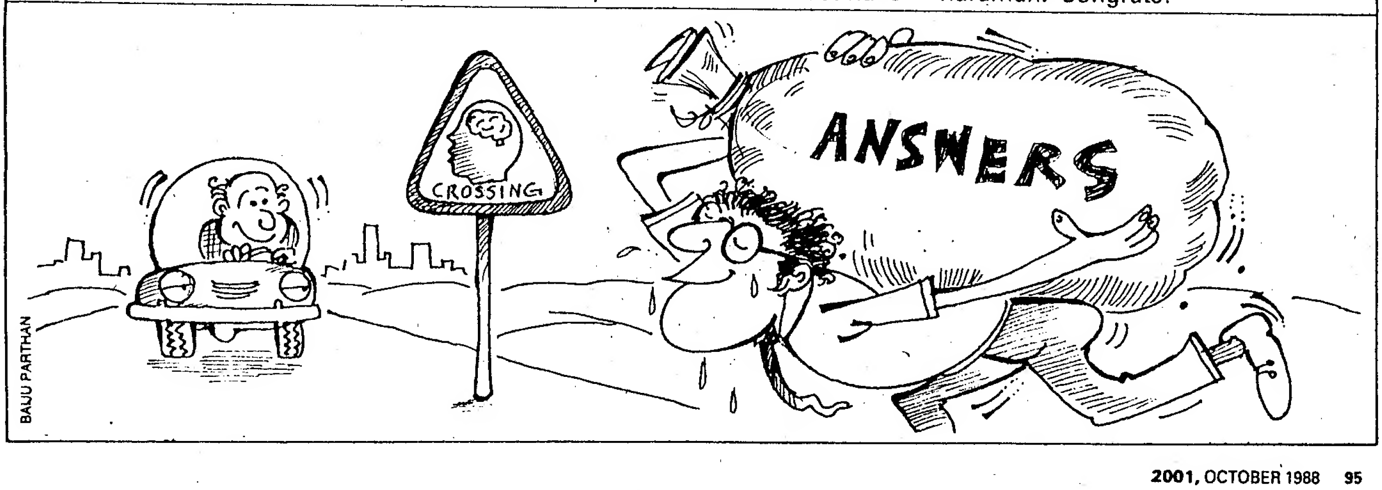
5. Suppose, n is the number of coconuts each man received when the pile was divided in the morning. Then there were (5n+1) coconuts in the final pile and $5[(5n+1)/4]+1=(25n+9)/4$ coconuts in the penultimate pile. So the last man who raided the pile must have taken (5n+1)/4 coconuts. His predecessor must have taken (25n+9)/16 coconuts. Reasoning in a similar manner, we find that the original pile must have contained

$$15n+11+\frac{265n+265}{1024}$$

coconuts. For this number to be an integer, 265 (n+1) must be divisible by 1024. Since 265 and 1024 are relatively prime, n+1 must be 1024 or n=1023. This gives the total number of coconuts to be 15621. (If we allow negative coconuts, −4 is also an answer. In general, the answers differ by $15625=5^6$.)

Correct answers are from Santosh Iyer, Ravindra N., A.K. Sahu, Sujata Bhagwat, Janakiraman, Kalyanaraman and Ram Prasad.

The best answers for the whole set were from Santosh Iyer and Kalyanaraman. Congrats!



PLAYTHEMES

PAIR FARE

T. PADMANABHAN

THERE was a good response to the March instalment. Here are the answers.

1. This problem can be solved by simple pairing. Each person in the party shook hands with either 0, 1, 2, 3, 4, 5, 6, 7 or 8 other people. Since there were nine people other than John at the party and since he got different answers from each one, we conclude that he must have received the above numbers 0 to 8 as answers.

Now consider the case of the person who shook hands with eight people. He (she) has shaken hands with everyone except his (her) spouse. This means that everybody in the party except that particular spouse must have shaken hands with at least one other person. So we are forced to conclude the following: The spouse of the person who shook hands with eight people must be the one who shook hands with zero persons.

The rest is plain sailing. By similar reasoning we can see that the person who shook hands with seven people is married to the person who shook hands with one person, the one who shook six hands is married to the person who shook two hands, five to three, and four to four. Here is the cinch. There has to be a couple in which both the husband and wife shook hands with four people. But when John asked the guests he did not get the answer four twice. Therefore that couple must be John and his wife. In other words, John's wife shook hands with four people.

Correct answers were sent by R. Ramachandran, Madras, Sudhir Pohankar, Indore, Dinesh Jain, Thimpu, Sharad Mathur, Kanpur, Jaideep Nandi, Calcutta, and Santosh Iyer, Jaipur.

2. The simplest way of tackling this problem is to represent the six people by the vertices of a hexagon. Connect any two vertices by a red line if the corresponding persons are strangers and by a green line if they are friends. All that we have to show now is that there will be at least one triangle with all sides red (three mutual strangers) or all sides green (three mutual friends). This can be seen as follows: Take some vertex A from which five lines emanate. Of these, at least three must be of the same colour, say, green (a similar argument works if it is red). Let these lines be AB, AC and AD. Consider the triangle BCD. If the sides of BCD are all red then we already have our answer. If not, at least one of them has to be green. This green side, together with two of the lines AB, AC or AD, will form a green triangle. Similar reasoning applies to all other cases, proving our claim.

Correct answers were sent by Santosh Iyer, R. Ramachandran, Sharad Mathur, and Kashikar, Nasik.

3. The three cards are: king of diamonds, queen of diamonds, and the queen of clubs. This problem turned out to be too easy. Everybody has sent in the correct answer.(I feel cheated when this happens!)

4. According to the description, the number of "bridge ends" is 4×4 plus 3×3 plus 2×2 plus 1×1 plus five more from the ends of the bridges that lead to the mainland. This gives a total of 35. But for any bridge system,the total number of "bridge ends" must be an even number. Hence the description is clearly impossible. The ancient wise men are right.

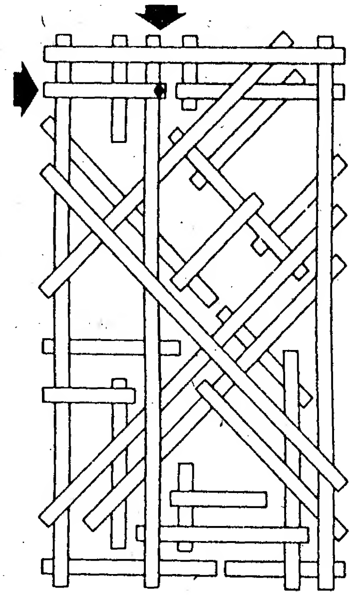
Correct answers were sent by Dinesh Jain, Sharad Mathur, Santosh Iyer, Jaideep Nandi and Soumitra Roy, Calcutta.

5. The correspondence between the sets is as follows: A is Y, B is X, C is W and D is Z.

Correct answers were from Soumitra Roy, Jaideep Nandi, Santosh Iyer, Prajnan Das, Calcutta, Sharad Mathur, Sudhir Pohankar, Dinesh Jain, and R. Ramachandran.

6. This problem was definitely easy and (again!) was solved by everybody. All we have to do is to take a fruit out of the box marked 'apples and oranges'. If it is an apple then that box has to be relabelled as 'apples'. The box originally marked 'oranges' should either contain only apples or a mixture. But since we have already identified the box containing only apples, this box must be relabelled 'apples and oranges'. Similar method applies if the fruit originally picked turns out to be an orange.

7 The highest point in the pile is near the left top edge as marked in the figure. This probably was the toughest question. Nobody got it right.



The best answers were from Santosh Iyer and Sharad Mathur. Congrats.

Four of the problems (3, 4, 5 and 7) are taken from the Mensa collection of puzzles devised by Steve Odell and others.

QUESTION BOX

THERE are some English alphabets to which words can be added to make new, meaningful words. For example, consider the letter 'I'; we can add the word 'dentification' to it obtaining the word 'Identification'. Challenge: Take each letter of the alphabet A,B,C ... etc. Find words for each one of them, which when added, make new words. I am sure it can be done for the whole alphabet. It would be wonderful if you can find: (a) the longest possible word for each alphabet, and (b) words which can be added to more than one letter of the alphabet. so as to minimize the total number of words. (For example, 'our' can he added to both s and h.) We will publish all the good solutions.

PLAYTHEMES

WHAT'S YOUR PROBLEM ?

T. PADMANABHAN

AFTER months and months of answers, it is time we looked at a set of fresh problems. Here we go:

1. A dog is lost in a square maze of corridors. At each intersection he chooses a direction at random and *proceeds* to the next intersection or manages to escape at one of the sides (see Fig. 1). His ordeal is over if he

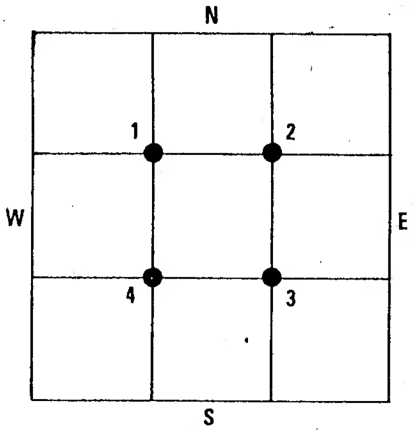


Fig. 1

reaches any one of the sides S, W, N or E. Tell me, what is the probability that the dog starting at the intersections 1, 2, 3 or 4 will exit at the south side?

2. Can you devise a chess problem having 'white to play and mate in three' from a given position irrespective of whether white is playing along up the board or down the board?

3. Here is a beautiful problem by Raymond Smullyan based on the good old 'lady or the tiger' puzzle.

The original story by Frank Stockton, as you will remember, is based on the following situation. A king makes a prisoner choose between two rooms, one of which contains a lady and the other a tiger. If he chooses the former he marries the lady. If he chooses the latter he gets eaten by the tiger. The king created by Smullyan did something more knotty. He gave the prisoner a choice of nine rooms! Each carried a sign on its door. Only one room contained the lady while each of the others contained either a tiger or was empty (that is, there is only one lady but there could be more than one tiger).

I Lady is in an odd-numbered room	II This room is empty	III Either sign V is right or sign VII is wrong
IV Sign I is wrong	V Either sign II or sign IV is right	VI Sign III is wrong
VII The lady is not in room I	VIII This room has a tiger and room IX is empty	IX This room has a tiger and VI is wrong

Fig. 2

The signs on the rooms are given in Fig. 2. The king added maliciously, "You see, the sign on the door of the room containing the lady is true; the signs on the doors of all the rooms containing the tigers are false, while those on the empty rooms can be either true or false." Since the prisoner had no chance with this meagre information, the king also condescended to tell him whether room VIII was empty or not. Miraculously enough, the prisoner could recognize the room containing the lady (and avoid it!). Can you?

4. Here is another classic which has appeared in many forms. There is a desert 800 km wide. An unlimited supply of petrol is available at one end but obviously there is nothing in the desert. A truck can carry enough petrol to go 500 km. Of course, the truck can take some petrol, keep it at some point in the desert and come back. On a later trip it can refuel itself using the petrol left in the desert. These refuelling stations can store as much petrol as needed. What is the optimum strategy for crossing the desert in the truck? (If you are able to solve this problem, attempt the general question with a desert X units wide and a truck capable of carrying petrol for Y units in a single trip.)

5. Given three circles which touch each other, one can add a fourth circle touching the other three. Two ways of doing this are shown in Fig. 3a and 3b. We start with three circles

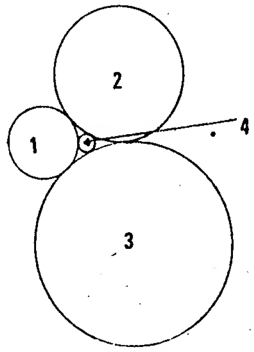


Fig. 3a

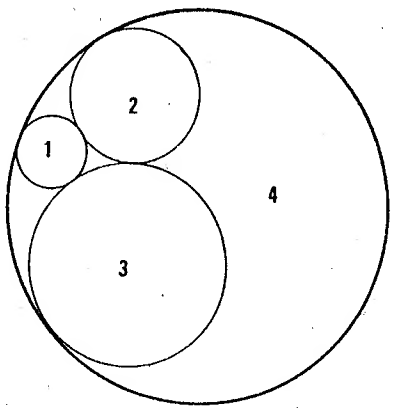


Fig. 3b

of radii 1, 2 and 3 units and add a fourth. Draw the circles as in Fig. 3a or 3b. What is the radius of the fourth circle?

6. In usual geometrical constructions one is allowed to use a compass and an unmarked ruler. Things can get tougher if we forbid the use of the ruler. Here's an old one (known as 'Napolean's problem'): Divide a circle whose centre is given into four equal arcs using the compass alone. In other words, find the four corners of a square which could be inscribed in this circle.

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SIMPLE, REALLY

KEEPING up with our task of clearing the backlog, we provide the answers to our June '88 issue.

1. Both slicings can be done. The cube is quite easy to slice and is shown in Fig. 1. The torus has to be sliced along a plane which passes through the centre and is tangential to the torus on the top and the bottom as shown in Fig. 2.

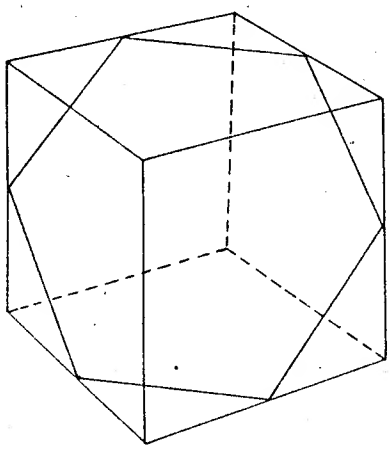

Fig. 1

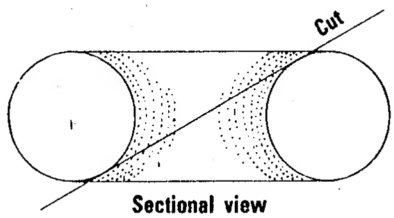

Fig. 2

The cube was solved correctly by V.K. Singhal, Lucknow; Santosh Iyer, Jaipur; R.J. Kadam, Pune; and N. Ramanathan and R. Charulatha, Ancheri (Kerala). No one cracked the doughnut.

2. Suppose all the angles of the triangle ABC have a measure less than 120°. Then we can find a point P inside it such that $\angle APB = \angle BPC = \angle CPA = 120°$. (The proof is very simple if you rotate the triangle by 60° about one of the vertices.) If the triangle has one angle larger than 120° then the vertex with that angle coincides with point P, which we have to find.

Correct answers were from Santosh Iyer, N. Ramanathan and R. Charulatha.

3. The simplest way to solve this problem is as follows: Reverse the steps of one of the drunkards. Now we have a single random walk of 100 steps. It is well known that the expected 'root-mean-square' distance in a random walk of N unit steps is $\sqrt{N}$. So here the answer is $\sqrt{100} = 10$ units. Santosh Iyer and A.J. Mitra, Calcutta, solved the problem by this method, while N. Ramanathan and R. Charulatha provided a detailed algebraic solution.

T.P.

FUN WITH MATHS

FERMATHS

C.P. NAZIR

PYTHAGORAS, the Greek philosopher and mathematician, formulated his famous theorem around 520 BC. The theorem, which states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides, that is, $a^2+b^2=c^2$, where c is the hypotenuse of a right-angled triangle with sides a, b, c, is now one of the basic theorems in geometry. In the year 1637, the French mathematician Pierre de Fermat proposed a theorem (also known as Fermat's Great Theorem) which said that there are no natural numbers x, y, z such that $x^n+y^n=z^n$, where n is a natural number greater than 2. Fermat did not provide any proof of his theorem, which went on to become one of the world's famous unsolved puzzles.

Mathematicians have attempted to verify the correctness of this theorem for more than three centuries, but without success. In March 1988, Yeichi Miyaoka, a Japanese professor claimed that he had solved this problem; but his claim was soon belied, and Fermat's theorem continues to elude solution.

The seeming simplicity of the theorem has goaded even ordinary math lovers to give a thought to this puzzle. While on it, the author came across an interesting equation, which says that in a right-angled triangle, the hypotenuse raised to the power n is equal to the sum of each of the remaining sides raised to the power n, multiplied by a co-efficient equal to the ratio of the hypotenuse and that side raised to the power n−2.

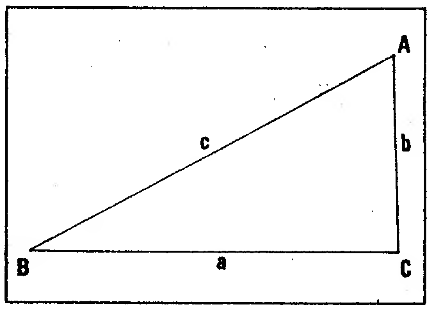

Fig. 1

The above relation between the sides of the triangle ABC, with a right angle at C (see Fig. 1) can also be expressed as

$c^n = x^{n-2} a^n + y^{n-2} b^n$

where $x = \frac{c}{a}$, $y = \frac{c}{b}$ and $n > 2$

The equation is derived as follows:

Given that $c^2 = a^2+b^2$ .... (1)

Multiplying both sides by c,

$c^3 = ca^2+cb^2$ .... (2)

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PLAYTHEMES

EVE TO ET

T. PADMANABHAN

IT has happened again! The questions posed in the April '88 issue produced such a massive response that it is not possible to include all the names in print. It is the fifth problem which seems to have troubled most people; but I must admit that it was solved by more people than I had expected. Here are the answers:

Talking of eves

Converting the recurring decimal into a fraction, we note that the quantity (TALK/9999) reduced to its lowest term must equal (EVE/DID). It follows that DID is a factor of 9999. Only three possibilities exist for DID: 101, 303 or 909. Of these, it is easy to see that 101 and 909 are ruled out. If DID=101 then we get TALK=99×EVE. Now EVE can't be 101 and anything bigger than 101 will not satisfy this equation. If DID is 909 then TALK=11×EVE. But then the last digit of TALK must be E which is not the case. So DID must be 303. Since EVE must be smaller than 303, E is 1 or 2. Of the 14 possibilities (121, 141, ... 292) only the number 242 fits the bill. Hence the answer is:

$$\frac{242}{303} = .79867986$$

Go ye and multiply

This one is easy to crack if you go about it the proper way. Since each digit is one of the set (2, 3, 5, 7) the only numbers that can go in the third and fourth rows belong to the following combinations: 3×775=2325; 5×555=2775; 5×755=3775 or 7×325=2275. This means that the two factors which are being multiplied must belong to the combinations 33×775, 55×555, 55×755 or 77×325. But only the first of these four possibilities leads to a fifth row made entirely of primes. Therefore the answer is:

```
  775×
   33
-----
 2325
2325
-----
25575
```

Just A,B,C...

The answer is 21978×4=87912. This can be arrived at by a series of steps requiring simple logic: (a) Since A multiplied by 4 is a single digit, A is 1 or 2. But since E×4 is even, a little experimentation shows that A must be 2. (b) Now the only numbers which on multiplication by 4 give a figure ending with 2 are 3 and 8. So E must be 3 or 8. But since A×4 cannot have two digits, it must be 8. Therefore E is 8. (c) Since 3 is carried over to D in the top line, B must be 1. (d) Now it is plain sailing. D must be 7 and C must be 9.

Logical deduction

One can show with some diligence and patience that this problem leads to a contradiction: We begin by noticing that (since the answer has no digit in the far left column) E must be 1. From the second column (from the right) it is clear that H must be either 0 or 9. Now we will show that neither of these possibilities are admissible for H.

If H is 9, then in the right-most column N must be zero and T must be 9. This is impossible because both H and T can't be 9. On the other hand, if H is 0, then in the fourth column from the right, I must be 0 or 1. Since 0 and 1 are represented by H and E, I cannot be either of these. Thus we run into a contradiction in all cases.

BAIJU PARTHAN

A large number of readers managed to crack these four problems. Those who got all the four right are T.N. Mukund, Bombay, K.P. Philipose, Wayanad, C. Thyagi, Banaras Hindu University, N. Swaminathan, Ernakulam, R. Charulata and N. Ramanathan, Trichur, Manoj Lala, Sholapur, T. V. Ramagopal, Madras, Santosh Iyer, Jaipur, T. N. Ramprasad, Madras, Mohamed Nissar, Anand and Himanshu, Bhopal, Devasundaram, Salem, M. L. Munvar, Bombay, V. Singhal, Lucknow.

The following got three answers right: C.G. Subramaniam (Madras), Gehanath Baral, Nepal, B. V. Raghavan, Salem, L. Satish and Rajanikanth, Bangalore.

Sorry about those who got two or less answers right. Lack of space forces me to give you a raw deal! I hope you will do better next time.

Provoking the ET

This problem was solved by only five readers. They obtained varying degrees of success. The message is really a work of art and shows how by a bit-by-bit process an elaborate correspondence can be set up. I explain below what the ET is expected to understand, line by line. (1) This line introduces the 24 symbols which are going to be used. It does not matter that the symbols are from English alphabets. The ET only needs to think of them as 24 symbols; it is not expected to know the English language. (If it did, communication wouldn't be such a big problem.) I am surprised that a few readers got stuck at this level. (2) This clue links the letters A, B, up to J with numbers from 1 to 10. (3) This line introduces the symbols K for 'plus' sign and L for 'equal to' sign. From line 2 the ET knows that B is 2. So it will interpret AKALB as 1+1=2 and will identify the meaning for K and L. (4) This introduces the minus sign by the symbol M, using a procedure similar to that of line 3. Similarly line 5 introduces 0. (6) This introduces the decimal nota-

PLAYTHEMES

UNSCRA(M)BBLING

T. PADMANABHAN

THE scrabbling question of July '88 produced several interesting responses. I don't think I really got a very high-score answer but the efforts were interesting.

The best answer I know of is due to Stephen Barr, the well-known mathematical puzzlist. His answer is as follows: he starts with the position as shown in Fig. 1 (we need only the lower half of the scrabble board). The player holds the letters G, I, L, O, T, Y and Z. In the position shown, most words are well known except probably the following: COCOS is the plural spelling preferred in the dictionaries. AR is a variant of ARE, a metric system measure; this word has found its way into several dictionaries. MEL is the Latin for honey, especially used by pharmacists, but given in the dictionaries without double parallel bars. If you grant all these, then the player can make the word TRANQUILIZINGLY along the bottom line of the board. The crucial trick, of course, is to cover three triple-word-score squares so that you get 3×3×3=27 times the original word score. This itself gives 27×37=999. The words DISJOINT and WAXY formed at the left and right extremes give another 3×16=48 and 3×17=51 points respectively. Add to this the bonus for using the seven letters—50 points—and you get a hefty score of 1148.

I must confess that the original configuration is somewhat shady, though probably legal in the strict sense. It all depends on which dictionary you are using (For example, the Random House edition I have does not give AR or MEL, but I know of other dictionaries which do. The word DOR, which means beetle, is also not found in all dictionaries. Because of these misgivings I enjoyed the solution by Anand Shrimali from Bhopal who fills the bottom row with the word PSYCHOPATHOLOGY. Unfortunately, this answer suffers from two defects. Anand Shrimali starts with a position containing — S – – – OP— THO—OG— and fills the blanks with the letters in hand, P, Y, C, H, A, L,Y, but does not bother to give detailed initial configuration. It will be quite tough to get word strings ending with THO and OG in adjacent columns. Incidentally, OP is a form of art and THO is the informal form of THOUGH; but I suspect OG should be disallowed. As far as I know, it is a philately term, 'O.G.' standing for Original Gum. The second defect with the solution is that several dictionaries do not have the main word PSYCHOPATHOLOGY. Well, this could be the study of psychopaths or a psychic way of pathological diagnosis. Probably we should invent this word if it doesn't exist. I hope Anand Shrimali will comment again. Anyway, the score is a stupendously high 1373 for this solution.

There was an attempt from Himanshu Shrimali, Bhopal, using the word PHOTOSYNTHESIZE. Unfortunately, the cross connections were all wrong.

Two others, K.F. Dhondy, Bombay, and Pragnan Das, Calcutta, used the word QUIXOTICALLY to score 446 and 455 points respectively. The configurations just before the word QUIXOTICALLY is introduced is shown in Fig. 2a and 2b. You can

D			B	E	R	A	T	E	D						
I						V				A					
S						O				S					
J						C	O	C	O	S					
O			P	I	K	E		H		A	R			W	
I	D	O	L		U	T		I		G	A	M	M	A	
N	O	V	A		D			L		A	I		E	X	
	R	A	N		U			I		I	N		L		

Fig. 1

		I			T			A	L	L					
		M			R					O					
		P	R	O	O	F				O					
		R			U		F			K					
		O		E	N	G	I	N	E	S					
		V			C		N								
		E			E		G								
						N	E	W							
							R								

Fig. 2a

					G	N	U
					O		
					O		
					S		
				R	E	N	T
				I			
B	R	I	N	G			
				H			A
				T	A	L	L
							L

Fig. 2b

easily figure out the scores. These solutions use really simple words.

The next best responses were from C.S. Venkataraman and Yogam Venkataraman, Bangalore, using the words MAGAZINE and PARAQUET cleverly to get 405 and 416 points. Their final configurations are shown in Fig. 3a and 3b.

The contest is still open. If you get anything more interesting, please do send. I conclude with an interesting variant in anagramming for you to worry about. Normally anagramming consists of rearranging the letters of a word to form just another word. It becomes really exciting when the transformation has an interesting relation to the original word. For instance, the word 'desperation' is an anagram to 'a rope ends it', 'panties' is an anagram to 'a step-in', and 'softheartedness' can be transformed into 'often shed tears'. Let us see how many more you can find having this quality.

						P	
				S	K	I	M
						P	A
					H	A	G
						L	A
							Z
						H	I
				M	O	A	N
	D	I	V	E	R	S	E

Fig. 3a

								D	
								I	
					S	K	I	M	P
								M	A
				C	L	O	V	E	R
							E	R	A
		F	U	S	E		T		Q
		A		L					U
		I		A			H	I	E
	G	R	A	V	A	M	E	N	T

Fig. 3b

FUN WITH MATHS

EASY GCD

P. K. MUKHERJEE

THE greatest common divisor (GCD) of any two numbers can be found by splitting up the numbers into their prime factors and then by taking the product of all those factors which are common to them. This procedure is in fact the one taught in schools. An alternative method for finding the GCD of numbers is provided by the Nichomachus or subtraction algorithm. This algorithm requires subtracting the smaller of the two numbers from the larger, then subtracting the remainder from the smaller number, further subtracting the new remainder from the previous one, and so on. The last nonzero remainder is the GCD of the numbers.

A more efficient algorithm was devised by the Greek mathematician Euclid of Alexandria who lived around 300 BC. In Euclid's algorithm, also known as division algorithm, instead of subtracting the smaller number from the larger and taking the difference, the procedure requires dividing the larger number by the smaller and taking the remainder.

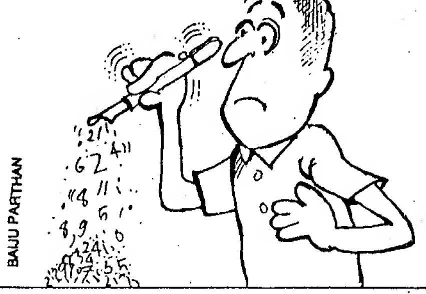
But mathematicians are never satisfied. *They are always in search of* devising better ways of performing calculations. The working of Euclid's *algorithm has been rendered more* efficient by evolving the method of 'least absolute remainders'. This requires taking the nearest multiple at every stage irrespective of whether it is greater or less than the number being divided.

Take an example of finding the GCD of rather large numbers, say, 3900 and 2520. Working with Euclid's algorithm results in the following steps:

$$3900=1\times2520+1380$$
$$2520=1\times1380+1140$$
$$1380=1\times1140+\ 240$$
$$1140=4\times\ 240+\ 180$$
$$240=1\times\ 180+\ \ 60$$
$$180=3\times\ \ 60+\ \ \ 0$$

But, on using the method of least absolute remainders, the number of steps is reduced from six to four:

$$3900=2\times2520-1140$$
$$2520=2\times1140+240$$
$$1140=5\times\ 240-60$$
$$240=4\times\ \ 60+0$$



For finding the GCD of more than two numbers, Euclid's algorithm has to be used more than once. Suppose it is required to find the GCD of a, b, and c. We first use the algorithm to find the GCD of the numbers a and b. Let (a, b)=r. Then we again apply the algorithm to find the GCD of r and c. Suppose (r,c)=s. So, s is the desired GCD of the numbers a, b, c.

The least common multiple (LCM) of two numbers can be related to their GCD. The LCM of two numbers equals their product divided by their GCD. So, the LCM of two numbers a and b is a×b/(a,b).

The methods for finding GCD and LCM of numbers have important applications to fractions. To reduce a fraction to its lowest term we divide both the numerator and the denominator by their GCD. To add and subtract fractions, we find the LCM of their respective denominators. In fact, it was the practical use of fractions in commercial calculations which led to the discovery of methods for calculating the GCD and LCM of numbers.

Besides having application in finding GCD of numbers, Euclid's algorithm has also *important uses in the* theory of Diaphantine equations. These (indeterminate) equations were named after the third-century Greek mathematician Diaphantus of Alexandria.

PLAYTHEMES

WORD FORWARD

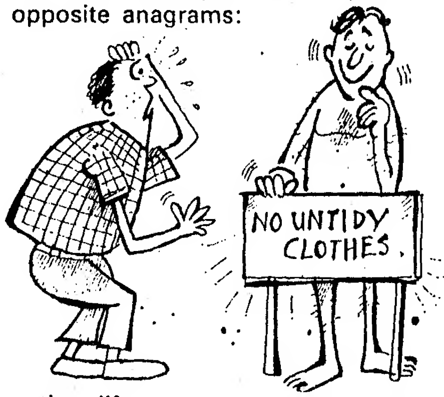
CONTINUING from where we left off last month, here are some more questions on the English language.

1. Some familiar words are also acronyms for things which may not be so familiar. Can you give the full form of the following acronyms? (a) PAKISTAN, (b) SCUBA, (c) WAVES.

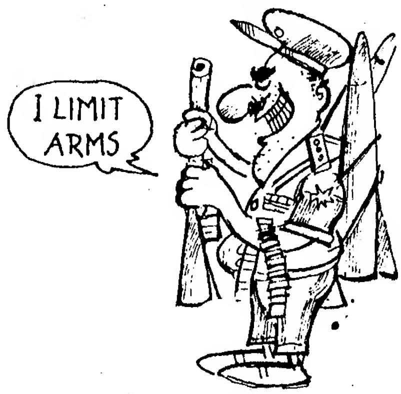
2. I'm sure you all know about a pride of lions or a pack of wolves. Now complete the following:
(a) a smack of...
(b) an unkindness of...
(c) a building of...
(d) a plague of...
(e) a mummuration of...
(f) a mustering of...
(g) a murder of...
(h) a school of...

T. PADMANABHAN

3. Anagramming is the art of rearranging the letters of a word or a phrase to form another meaningful word or phrase. The ingenuity lies in making the new phrase mean exactly the same or quite the opposite. Here are some examples for simple and opposite anagrams:



a shop-lifter	has to pilfer
the eyes	they see
a decimal point	I'm a dot in place
the countryside	no city dust here
the detectives	detect thieves
the nudist colony	no untidy clothes



Some opposite anagrams are as follows,:

misfortune	it's more fun
violence	nice love
evangelists	evil's agents
militarism	I limit arms

Now can you get me some better ones? In particular, can you think of some 'antigrams' for the following?
(a) considerate
(b) real fun
(c) care is noted

Illustrations: Baiju Parthan

4. Now, some more posers on vowels.
(a) There are three English words that contain the vowels a, e, i, o, u once and only once in their reverse order, that is, as u, o, i, e, a. Find them.
(b) There is a common nine-letter English word which contains only one vowel. Which is it?
(c) Find a seven-letter word, again not uncommon, which does not have a, e, i, o or u.
(d) There is a 15-letter word which has only one vowel that is repeated five times. Can you spot it?

5. Finally, here are some more assorted sets going in the same vein.
(a) Name two words each of which is only eight letters long, but contain the first six letters of the alphabet.
(b) There is a word which runs to 15 letters but can be printed without any ascenders like, say d, or descenders, like q. Don't try to be smart and write the word in capital letters.
(c) There are two 17-letter words which contain the same 17 letters. Do you know them?

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PLAYTHEMES

SCIENTIFICALLY SPEAKING

T.PADMANABHAN

THE science problems of September '88 produced interesting responses. I'll hold some of the questions open so that readers can send in more answers. Here are the 'solutions', but don't believe them blindly!

1. Incredible though it might seem, there is one effect which makes the quantity of coffee in a cup on the Moon (filled up to a given mark in the rim) increase. The surface of a liquid is not really flat but is approximately spherical and parallel to the Earth's surface. Since the Moon is smaller than the Earth, its surface is more curved. The surface of coffee in the mug on the Moon is also more curved and thus the mug holds more coffee while on the Moon. Needless to say, the increase is negligibly small, but does exist in principle.

The trouble with harping on such 'negligible' effects is that there may be other comparably small effects which produce opposite results. In this case I hope there aren't any. But if you find one, please do write. (In particular, you might like to worry about the fact that coffee, like any other liquid, is compressible. Since gravitational pull will be larger on the Earth, the lower layers of the coffee will be slightly more compressed on the Earth than on the Moon. Can this affect the conclusion?)

None of the readers came up with the curvature effects clue; the best analysis was by D. Krishna Warrier, Trivandrum, and N. Ramanathan, Trichur.

2. If you shine light of different colours from independent sources on to the stage, then you are very likely to get a whitish tinge. In this respect, it is not like mixing paints. Incidentally, if you use blue and yellow filters in the same lamp, you will get something like green light only because these filters transmit light over a band of frequencies. Suppose you have a strictly monochromatic filter, that is, a filter which will transmit light of onlv a fixed frequency f_1. If you place it along with another filter of frequency f_2, then you will get no transmitted light at all. A normal blue filter will transmit most of the blue light; it will also partially transmit the neighbouring colours, say, a bit of indigo and green. Similarly, a normal yellow filter will transmit most of yellow and probably a bit of green and orange. So when the filters are used together green light has the best chance of getting through.

3. The room should get hotter. Remember that a refrigerator, while working, consumes power in order to cool the contents. Normally that heat would be given out at the back of the refrigerator and its inside will be kept cool. Ultimately the heat given out at the back will increase the temperature of the surrounding. In the present problem, the same situation will occur with a vengeance. The open fridge takes in some heat from the room and gives out more heat at the back, the extra amount coming from the energy consumed by the fridge. The heat given out at the back recirculates in the room. So, eventually the fridge is acting as a room heater.

I am sure there will be some dissenting voices regarding this problem. I will keep the discussion open.

The above answer was given by N. Ramanathan, Krishna Warrier, and Niraj A. Shah, Bombay.

4. (i) Madras is farther from the centre of the Earth than Bombay. Remember that the Earth is not a sphere but is slightly flattened at the poles. (ii) The centrifugal force at Madras is larger than that at Bombay. Both these effects will reduce your weight in Madras compared to Bombay. But again, alas, by a negligible amount.

Correct answers were from Manish Bahuguna, New Delhi, Krishna Warrier, Ramanathan, and Niraj Shah.

5. The white snow causes most of the Sun's heat to be reflected. If black soot settles down on the polar region, it will absorb more heat and will start the snow melting. So broadly speaking, a greater amount of the Sun's energy would be absorbed by the Earth making everything warmer.

BAIJU PARTHAN

There is, however, one significant difference between the icecaps of the North Pole and the South Pole. Most of the ice in the North Pole floats in water. The ice in the South Pole is mostly on land. As a result, the melting of the South Polar caps would cause a rise in the sea level and a certain amount of flooding.

Almost correct answers were given by Niraj Shah, P.V. Subramanian, Hyderabad, and Manish Bahuguna.

6. Yes, in principle this scheme will work, as noted by Krishna Warrier, N. Ramanathan, and P.V. Subramanian.

7. The time taken for one swing of the pendulum increases with the 'effective length' of the pendulum. The effective length here is the distance from the point of suspension to the centre of gravity of the bob of the pendulum. As the water leaks, the centre of gravity will come down increasing the effective length. Thus, the time for swing will go up initially. This would eventually reach a maximum after which the time for swing would decrease. Eventually all the water would have gone bringing the centre of gravity back to the centre of the bob. Now the time for swing would be back to normal.

I am surprised nobody really answered this question right. Everyone seems to have worried only about the change of mass of the pendulum bob but not about the change in the position of the centre of gravity. The answer closest to truth was from N.Ramanathan.

8. This problem is basically a play of words. If a motor is connected to a load, the energy is ultimately supplied by the electrical power source running the motor rather than by the magnets. (Of course, a motor will rotate faster if the current through the coils is kept constant and the magnetic field is increased.)

Correct answers were from Krishna Warrier, Ramanathan, and Niraj Shah.

9. Everybody answered this question right (the boat doesn't move), in spite of my efforts to confuse the issue!

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PLAYTHEMES

MORE TRICKS ON THE CARDS

T.PADMANABHAN

CLEARING up a lot of backlog leaves me free to build a new one. So I'll start right away.

1. What is the fewest number of bids that can precede play in a game of Contract Bridge?

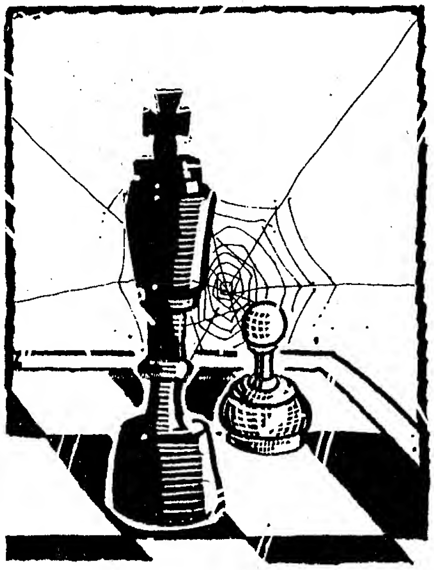
What is the number of bids preceding play in the longest possible legal auction in Bridge?

While counting such things, can you tell me what is the maximum number of moves possible in the longest (and probably dumbest) legal game of chess? (You may think a game can go on indefinitely if you don't know *all* the rules).

2. Your friend, who is a bit of a gambler, offers you the following deal: "I have put three cards inside this box. One card is red on both sides, another is white on both sides and the last one is white on one side and red on the other. I'll take one of them out at random and place it on the table. Here, Ha! the top side is red. Now tell me, what is the probability that the other side is also red?"

I agree the answer is fairly easy. But credit will go to the reader who comes up with the shortest, neatest and most intuitive proof for the right answer.

One of the most irritating brain teasers making the rounds in the seventies (I think SCIENCE TODAY is also guilty of publishing it!) was the following: "Find the rule according to which the following sequence is constructed — O, T, T, F, F, S, S, E, N, T." They happen to be the first letters of One, Two, Three... etc. I have a more irritating version of this question: How is the following sequence formed — T, T, T, F, F, S, S, E, N, O? (Of course, I have merely interchanged the first and the last letters of the original sequence — but that's not the answer.) Have fun.



BAIJU PARTHAN

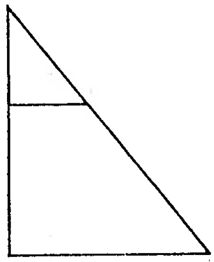


Fig. 1

4. Fig. 1 shows two Pythagorean triangles, one 'inside' the other. You are required to construct this figure by placing matchsticks. You can't, of course, break a match or do any other similar mischief. What is the minimum number of sticks you will require?

5. The Queen wants to go from c6 to f3 in a chess board (Okay, if you're still old fashioned read it as from QB6 to KB3.) Being somewhat jealous of the knight, she also wants to make it after visiting each of the sixty-four squares once and only once. And being more regal, she wants to do this with the fewest number of turns. (Of course, the queen can move horizontally, vertically and diagonally.) Can you help plan her tour?

6. Name the southernmost points of Europe, Africa and South America.

While at it, what is the southernmost point in India?

FUN WITH MATHS

NINE TO NINE

P.K. MUKHERJEE

CONTINUING our adventure with numbers, we focus on the peculiarities of numbers which have nine non-repititive digits. If the digits of these numbers be arranged in descending and ascending orders respectively and their difference taken, the resultant nine-digit numbers posses the same nine digits. Take an example, that of 875194326 and carry out the above procedure. We get a result 864197532:

$$\begin{array}{r} 987654321 \\ -123456789 \\ \hline 864197532 \end{array}$$

The same operations carried out on the numbers 917654321, 982654321, 987354321 and 987644321 will give an identical result, namely, 864197532. Note that in all these numbers one of the digits 5, 6, 7, or 8 is missing and its place is occupied by the digit 4, 3, 2, or 1 such that the sum of the missing digit and that occupying its place is always 9. This indeed is a very interesting and curious coincidence.

Now form another nine-digit number by using each of the digits 4, 5 and 9 thrice. Arranging the digits of the resulting number in the descending and

PROBLEM SOLVING ... WELL ALMOST

T.PADMANABHAN

THE questions posed in the February instalment had a fair response. Here are the answers.

1. The quickest way to solve this problem is to make full use of the symmetries. Let P_1, P_2, P_3, and P_4 denote the probabilities for the dog to exit at the south side starting from intersections 1, 2, 3 or 4 respectively. From symmetry, $P_1=P_2$ and $P_3=P_4$; also if the dog is *randomly* placed at an intersection, there is a ¼ chance of it exiting from the south, implying $(P_1+P_2+P_3+P_4)/4=1/4$. Now suppose the dog starts at intersection 1 and moves one unit to intersection 2 or 3 (each has a probability 1/4), and finally exits at the south. By invoking the usual law for combining the probabilities, we get $P_1=P_2/4+P_3/4$. These relations can now be solved to give $P_1=P_2=1/8$, and $P_3=P_4=3/8$. (This problem was originally proposed by a reader in the puzzle corner run by A.J. Gottlieb in *Technology Review.*)

Correct answers were received from Santosh Iyer; Jaipur, N.Ramanathan, Kuwait, and Alka Jain, Thimpu.

2. Nobody could find a solution to this problem! Several readers have made the 'obvious' observation: If the position contains no pawns then it does not matter whether white is playing up or down the board. (This is not quite true if we have to consider the possibility of castling but I will let it pass.)

To settle it once and for all, let me rephrase the problem: construct a chess position, *containing movable pawns,* in which white can play and mate in three irrespective of whether white is playing up or down the board. Come now, no more excuses. I want an explicit solution. This question is still kept open.

3. The Raymond Smullyan problem on the lady or the tiger produced interesting responses. (Unfortunately, some readers have misunderstood the problem.) The following is probably the shortest line of reasoning: (i) If the king declared room VIII as empty, the prisoner could not have solved the problem; so he must have said that room VIII was not empty. (ii) If the lady was in room VIII, the sign VIII must be true, leading to a contradiction; so room VIII must contain the tiger. (iii) Sign VIII must be false since the room has a tiger; the only way for this to happen is for room IX not to be empty. (iv) Room IX cannot have the lady (for the sign will then be true), and must have the tiger. So sign IX is false, sign VI is true, and sign III is false. (v) Rest of the reasoning is fairly trivial and leads to the unique solution that the lady is in room VII.

Correct answers were from Santosh Iyer and N.Ramanathan. Several other readers sent in a partial analysis of the problem.

4. Let us consider the distance of 500 miles as one 'unit' and the petrol needed for this distance as 'full load'. It is easy to see that with two full loads, the truck can go up to $(1+1/3)$ units. This is done as follows: (a) start with full load, go up to ⅓ of the unit, deposit ⅓ of the load there and return; (b) start with another full load, stop at the ⅓ unit point, pick up the ⅓ load left there earlier, and travel forward another one unit, thereby reaching a distance of $(1+1/3)$ units from the base. Similarly, with three full loads we can travel up to $(1+1/3+1/5)$ units. We make two full load trips to stock $(3/5)+(3/5)=6/5$ load of petrol at a distance of $(1/5)$ from the base. In the third trip, on arriving at the ⅕ unit point we will have ⅘ load in the truck and 6/5 load in stock. Together, this is enough to go for a further distance of $(1+1/3)$ units starting from the first base. (Of course, we will have to set up another stock point) as explained before.

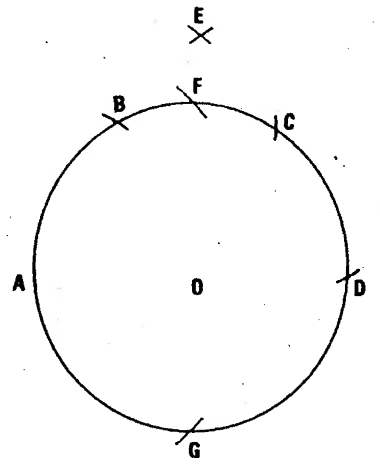
It is now clear that with k full loads of petrol we can go up to $1 + (1/3) + (1/5) + \ldots [1/(2k-1)]$ units. For the figures given in the problem, we see that three full loads will only take us up to 766 ⅔ miles. We can cover this extra 33⅓ mile by setting up another storage point at this distance from the start. A total of 16 trips and 3 7/15 load of petrol will do the job.

The following readers have sent answers to this problem: N.Ramanathan, Kuwait, R.S. Kumar, Salem, Alka Jain, Santosh Iyer, Asit Samanta, Bombay, and Ashim Kumar Kar, Durgapur. The best analysis was from N.Ramanathan and Santosh Iyer.

5. There is a simple formula connecting the radii of the four circles which touch each other. If a, b, c and d are the *reciprocals* of the radii, then we must satisfy the condition $2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$. Using this formula, it is easy to find the radii of the two circles to be (6/23) units and 6 units.

Correct answers were from Ashim Kumar Kar, Santosh Iyer, Alka Jain, and N.Ramanathan.

6. Several different solutions are available to this problem. I give here the one due to the famous puzzlist H.E. Dudney: (i) With the compass set to the radius of the circle, mark spots B, C and D starting from any point A. (ii) Draw two arcs with centres A and D and radius AC, intersecting at E. (iii) With centre A and radius OE, draw the arc that cuts the original circle at F and G. (vi) A, F, G, D are the required points (see figure below).



Similar (correct) constructions were sent by N.Ramanathan, Alka Jain, Santosh Iyer and Ashim Kumar Kar.

The most complete set of answers to this instalment of problems was from N.Ramanathan and Santosh Iyer. Congrats!

Don't forget that question 2 is still open.

PLAYTHEMES

FABULOUS FABLES

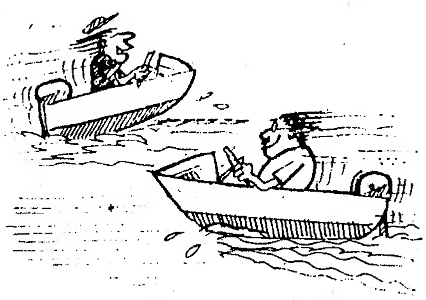
T. PADMANABHAN

SAM LOYD (1841-1911) was one of the greatest American puzzlists. For more than 50 years, his puzzles appeared in innumerable magazines and newspapers, and a few of his mechanical puzzles had become a national craze. Here is a small sample of Loyd puzzles for you to try.

1. A farmer possesses a field in the shape of a right-angled triangle. He wants to know how many rails of equal length would be required to enclose this field if one of the sides was 47 rails long. (That is, find a right-angled triangle with integral sides, one side of which is 47.)

2. An ancient problem involving a marching army is the following: an army column, 50 miles long, is marching at a constant rate. A courier starts at the rear of the army, rides forward to deliver a message to the front, and returns to his position. In this time, the army advances by exactly fifty miles. How far did the courier travel?

That was just a warm-up for you. Sam Loyd adds a further twist to the problem. Suppose the army was in square formation, 50 miles long and 50 miles wide, marching forward at a constant rate. The courier starts at the middle of the rear end and makes a complete circuit around the army and comes back to his starting point. The courier's speed is constant and he completes the circuit just as the army advances by 50 miles. How far does the courier travel? (The courier is keeping himself as close to the army as possible.)

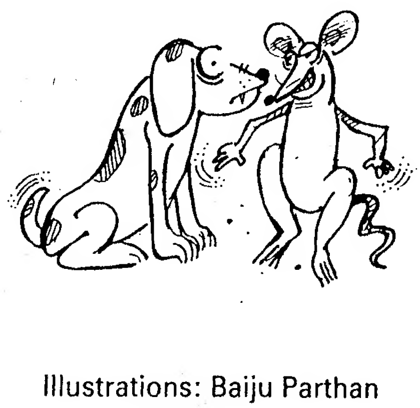


3. Two boats start towards each other at the same instant from the opposite banks of the river. One is faster than the other and the two meet at a point 720 yards from the nearer shore. After reaching the opposite banks, each boat remains at the dock for ten minutes and then starts back. They now meet at a point 400 yards from the other shore. How wide is the river? (The speed of each boat remains constant.)

It is not difficult to solve this problem using algebra. But with the right approach, it can be done in the head.

4. Here is another beautiful creation of Sam Loyd involving Diaphontine analysis: a merchant bought a certain number of puppies and half that many pairs of rats. He paid two bucks each for the puppies and the same price for each pair of rats. He then sold the animals at a price which was ten per cent higher than what he paid for them. After he sold off all but seven animals, he found that he had got back the amount he had originally invested.

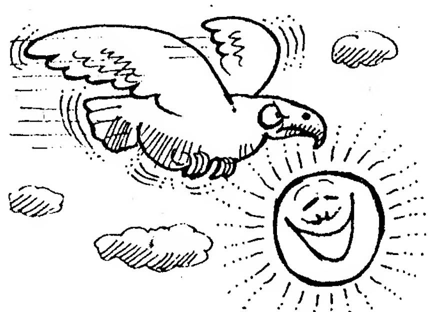
What are the seven animals worth at the merchant's retail prices?



Illustrations: Baiju Parthan

5. In the advertisements displaying the face of a clock, the hands are usually shown indicating an approximate time of ten minutes past ten or twenty minutes past eight. Assuming that the hands are at an equal distance from the 12 hour and 6 hour mark respectively, what is the exact time shown by the clock?

6. Among Aesop's fables is the story of the ambitious eagle which chases the Sun. Every morning it will fly



eastward (always facing the Sun) until it is noon; then as the Sun begins to move westward, the eagle will reverse direction and will fly west. "Just as the sun disappears, the eagle will be back where it started," concludes Aesop.

Aesop's mathematics is a bit shady. The afternoon flight will be for a longer duration carrying the eagle a little westward each day. Suppose the eagle starts from the dome of the capital at Washington (where the Earth's circumference is 19,500 miles) and flies at a height from the Earth's surface which does not materially affect the distance. Assume further that, each day it ends its flight 500 miles west of the starting point.

How many 24-hour days will have elapsed at Washington, from the time the eagle starts its flight till it ends it, having completed one westward circuit of the Earth?

PLAYTHEMES

MORE UNSCRA(M)BBLING

T.PADMANABHAN

RESPONSE for the scrabbling questions, which were answered in March '89, continue to pour in. So, as promised, here's more of the same.

C.S. Venkataraman, Bangalore, writes, "Looking into the *Shorter Oxford Dictionary* I spotted the word 'SESQUIOXIDE'—a chemical salt having two atoms of oxygen to three atoms of another substance. Well then, SESQUIOXIDIZING must be the process of making this salt. (That's a bit of extrapolation but let's be sportive and see where it takes us—T.P.) So, our word has 15 letters. Q and Z fall on double word squares and X sits smugly on a triple word square." He goes on to produce the configuration shown in Fig.1, giving a total score of 1935.

S	E	S	Q	U	I	O	X	I	D	I	Z	I	N	G
W	A	T		N	A	R	R	O	W		A	R	E	A
I	R	E			M		A	N	E		N	E		S
T		A			B		D		L		Y			T
C	A	K	E				I		L					R
H							A						J	O
I		D					T						E	N
N		O					E				U	P	T	O
G	O	G					D					U		M
												R		I
												S		C
											B	E	T	A
														L
														L
														Y

Fig. 1

Venkataraman's attempt deserves credit, but it uses several combinations which, I believe, are objectionable (for instance, it uses JO, EN, NE, WAT and XRADIATED). Probably this could be improved upon. The gastronomical sesquioxidizing sounds too tempting a preposition to be squandered away.

Discussing the PSYCHOPATHOLOGICAL solution of Anand Shrimali in March, I had raised two objections. Anand has written back clearing one of them. The keyword psychopathology does appear in *The Concise Oxford Dictionary and Reader's Digest Dictionary.* That makes it quite legitimate. Anand has sent a series of configurations which can be completed with psychopathology to produce the final result. Unfortunately, most of these positions contain cross connections which are not allowed. (For example, the rules of scrabble forbid the presence of a vertical column of letters, OVERRUNAURORA, on the board without any gap in the middle.) One of the 'better' configurations, shown in Fig. 2, contains cross connections involving OL (which I think is the medical short form for *oculus laevus,* the left eye, but usually spelled in capitals as O.L.), O.G. (which I've already objected to), and RB (which, if it is the symbol for the element rubidium, is again spelled with capital R). So I still find the psychopathology solution incomplete.

							A							
							U							
							N							
						S	T	A	C	K				
						T			L			T		
						E			O	L		R	B	
	A	D	V	I	S	E		A	T	E	L	I	E	R
	S				O	P		T	H	O		O	G	

Fig. 2 Lower half of the board

At the end of the March column, I had asked for interesting variations on anagrams. Here are a few interesting ones I got from Hirdesh Rohatgi, Delhi. 'Catalogues' is an anagram for 'got as a clue', 'elegant' can be transformed into 'neat leg', 'matrimony' can read as 'into my arm', 'revolution' as 'to love ruin' and 'astronomers' as 'no more stars'. That last example is an interesting one, since 'astronomers' is also an anagram to 'moonstarers'. Anyone out there knowing other words with double anagrams?

Long back, in November '88 in fact, I had set a task for readers to find word strings which can be attached to the alphabets to produce new words. Several entries were received, the best being that (given below) from R. Poonguzhali, Tanjavur.

A	chromatically	→ achromatically
B	racket	→ bracket
C	arouse	→ carouse
D	evolution	→ devolution
E	numeration	→ enumeration
F	rigidity	→ frigidity
G	ratification	→ gratification
H	arbour	→ harbour
I	deal	→ ideal
J	angle	→ jangle
K	nicker	→ knicker
L	imitation	→ limitation
M	uttering	→ muttering
N	either	→ neither
O	mission	→ omission
P	resumption	→ presumption
R	evolutionary	→ revolutionary
S	laughter	→ slaughter
T	reasonably	→ treasonably
U	sage	→ usage
V	alley	→ valley
W	right	→ wright
Y	arrow	→ yarrow
Z	one	→ zone

As you can see, the solution is far from perfect; there is no suggestion for Q and X. But the selection of words seems to be rather simple. For comparison, I give below the solution suggested by Allan Gottlieb in his puzzle corner of *Technology Review:* aquintocubitalism, blithesomeness, chemotherapeutics, demulsification, emotionlessness, frightfulness, gastrophotographics, hedriophthalmons, identification, japaconitine, kineasthetic, limitableness, methylhydrocupreine, neopaleozoic, oesophagostenosis, premosrepresentation, quinta, revolutionally, selectiveness, treasonableness, utopographer, vindictiveness, whenceforward, xanthosiderite, yourselves and zoosporiferons! (Please don't write asking for meanings or the names of dictionaries in which they can be found.)

Lastly, remember the tea cup problem posed in September '88 and discussed in May '89? Krishna Warrier, Trivandrum, wrote back, easily demonstrating that the effect due to compressibility is far larger than that due to the change in the radius of curvature.

FUN WITH MATHS

RECYCLING

VIR NARAIN

NUMBERS sometimes behave in the oddest ways. Recently, I (re)discovered, more or less by accident, a rather bizarre phenomenon which I shall call 'cyclic divisibility'. Take, for example, the number 169534. This number is divisible by 37. Not much to that, you might say. *Many* numbers are divisible by 37. The peculiarity of our number is that 37 divides its digits taken cyclically from any starting point. For instance, 695341/37=18793, 534169/37=14437; etc. But take the same digits in the reverse (anticlockwise) order, say 435961, and no number formed thus is divisible by 37.

Now consider the number 672549. This is cyclically divisible by 37 in both directions, clockwise and anticlockwise. So 945276/37=25548, 527694/37=14262 etc. It now transpires that: 1) Any six-digit multiple of 37 is cyclically divisible by it, at least in the clockwise direction. 2) Any six-digit number of the form a, b, c, d, e, f so that a+d=b+e=c+f is cyclically divisible in both directions by 37. We will call this full cyclic divisibility.

Now here is a curious coincidence. If you take the value of e (2.718281), the first six digits after the decimal place form a number possessing full cyclic divisibility by 37 and, as we shall see later, there is a similar coincidence if you take 22/7—an approximation of the value of π.

Apart from 37, some other numbers are also involved in cyclic divisibility, like 11, 13, 27, 33, 39, etc but only some of them in both directions.

But the last word in cyclic divisibility is the case of the number 142857 (if you try to get the approximate value of π by dividing 22 by 7, this is the number formed by the first six digits after the decimal point). This number 142857 is not only cyclically divisible by 3, 9, 11, 13, 27, 33, 37 and 39, it is also cyclically divisible by itself!

This is surely a unique phenomenon for a number with non-repetitive digits. What is more, the number formed by the cyclic division of 142857 by itself—132645—is again cyclically divisible by 37 in both directions. And the numbers yielded by such cyclic division, arranged radially, form further circles of numbers with the cyclic property (see Fig. 1 and Fig. 2).

The six digits in each circle form numbers which again show full cyclic divisibility by 37 (observe that the sum of the digits along each of the three diameters is the same). Thus the cyclic division of just one number (132645) has yielded ten more numbers (five in each direction) with the same property; and each of these numbers in turn would yield another generation (though not necessarily five each way) of such numbers. If this process were continued long enough, one would end up with the number 999999. Incidentally 142857×7=999999.

All this had basically to do with two numbers, namely 142857 and 37, quite a remarkable pair. But there must be many more such numbers and curious properties. Obviously, there is a great deal of (not necessarily new) territory to explore.

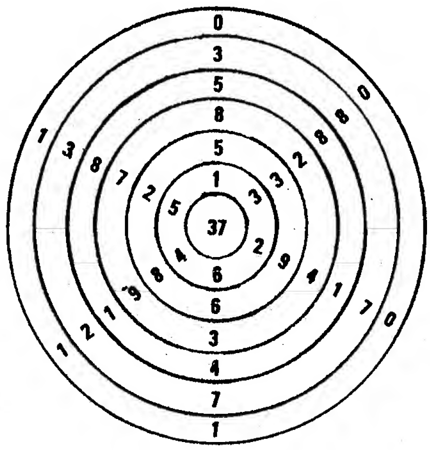


Fig. 1 Clockwise wheel: 132645/37

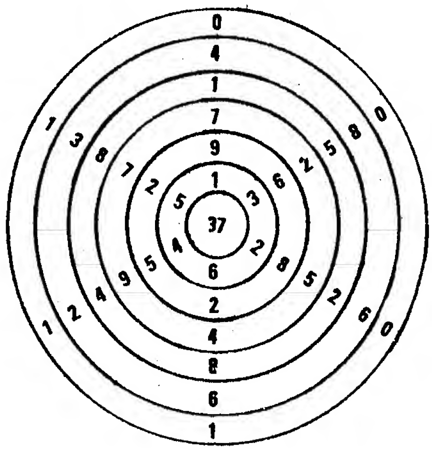


Fig. 2 Anti-clockwise wheel: 132645/37

MAGIC AND ANTI-MAGIC

V.R.MURALIDHARAN

IN the February '88 issue of SCIENCE TODAY we defined a magic number M as one where (M+1) is a perfect square and (M+2) is twice a perfect square. Going a step further, a generalized magic number M of order p is such that (M+1) is a perfect square and (M+p) is p times a perfect square.

Some examples are:

Order 2 $M=48$ $48+1=1\times7^2, 48+2=2\times5^2$
Order 3 $M=24$ $24+1=1\times5^2, 24+3=3\times3^2$
Order 5 $M=840$ $840+1=1\times29^2, 840+5=5\times13^2$
Order 6 $M=288$ $288+1=1\times17^2, 288+6=6\times7^2$
Order 7 $M=840$ $840+1=1\times29^2, 840+7=7\times11^2$
Order 8 $M=120$ $120+1=1\times11^2, 120+8=8\times4^2$

For all values of p other than a perfect. square an infinite number of values of M exist. They can be found (without trial and error) by the iterative formulae.

$x_{n+1}=ax_n+pby_n$
$y_{n+1}=bx_n+ay_n$
$M_{n+1}=x^2_{n+1}-1=2y^2_{n+1}-2,$

where a and b are integers satisfying the equation $a^2=pb^2+1$ and $x_1=y_1=1, M_1=0$.

We can calculate M of order 2. Since $a=3, b=2$ satisfy the equation $a^2=2b^2+1$, the iterative formulae for order 2 are

$x_{n+1}=3x_n+4y_n$
$y_{n+1}=2x_n+3y_n$
$M_{n+1}=x^2-1$

where $x_1=1, y_1=1, x_2=3\times1+4\times7, y_2=2\times1+3\times1=5$ and $M_{n+1}=7^2-1=48$.

For $x_2=7, y_2=5$ we get

$x_3=3\times7+4\times5=41$
$y_3=2\times7+3\times5=29$, and
$M_{n+1}=41^2-1=1680$

This process yields successively M=0, 48, 1680, 57120, 1940448, 65918160, 2239277040, 76069501248 etc. We can also obtain large M of order 2 by using higher values of a and b. For example, a=577, b=408 satisfies $a^2=2b^2+1$ Using x=8119, y=5741 we find that $x_{n+1}=577\times8119+2\times408\times5741=9369318$ $y_{n+1}=408\times8119+577\times5741=6625109$ and corresponding $M=9369318^2-1=2\times6625109^2-2=87784119785123$

The same number M may be (and usually is) a magic number for many values of p. Consider the number 1680. It is magic for orders 2, 10, 14, 21, 35, 48, 70, 112, 210, and 560 for

$1680+10=10\times13^2$
$1680+21=21\times9^2$
$1680+35=35\times7^2$
$1680+210=210\times3^2$, and so on.

Generally if p and q are two integers and q^2-1 is exactly divisible by p^2-1 giving a quotient of a, then putting $N+a=ap^2, N+1=q^2$ we get a magic number of order a.

In contrast, we can define an 'anti-magic' number by subtraction.
If $M'-1=P^2$ and $M'-a=aq^2$ where a, p and q are integers then M' is an anti-magic number of order a.

Some examples of order 2, where M takes values 10, 290, and 9802 respectively.

$10-1=1\times3^2$, $290-1=1\times17^2$,
$10-2=2\times2^2$, $290-2=2\times12^2$,
$9802-1=1\times99^2$
$9802-2=2\times70^2$

Anti-magic numbers do not exist for all values of a. They exist for a particular order a, if and only if two numbers, say, k and m (both less than a) can be found such that $ka-1=m^2$. If one solution to a particular order can be found then others (an infinite number of them) can be derived by analogous formulae:

$x_{n+1}=ax_n+bpy_n$
$y_{n+1}=bx_n+ay_n$
$M'_{n+1}=x^2_{n+1}+1=py^2_{n+1}+p$

where p is the order, a and b are two integers satisfying the condition $a^2=pb^2+1$ and x_n and y_n are two integers satisfying $x^2_n-py^2_n=p-1$. For example, consider order 13. Since $5\times13-1=8^2$, anti-magic numbers of order 13 exist. Since $65-1=1\times8^2$ and $65-13=13\times2^2$, 65 is a magic number of order 13 and $x_n=8, y_n=2$. The equation $a^2=13p^2+1$ has a solution a=649, b=180. Therefore,

$x_{n+1}=649\times8+13\times2\times180=11672$
$y_{n+1}=180\times8+649\times2=2738$
$M'_{n+1}=(11672)^2+1=13\times(2738)^2+13=136235585$

'Amicable' magic numbers are also possible. If N is a number such that $N+a=bp^2$ and $N+b=ap^2$ where a, b, p and q are integers, then N is an amicable magic number. For instance, for values of N like 7, 51, 115,

$7+1=2\times2^2, 51+1=13^2\times2^2$
$7+2=1\times3^2, 51+13=1\times8^2$
$115+1=29\times2^2$
$115+29=1\times12^2$

BRAIN TEASER

The kindly duke decided to give three of his servants a piece of land each as a long service award. He gave each a length of wire 144 yards long and some posts and instructed the first to mark out his land in the form of an equilateral triangle, the second in the form of a circle and the third in the form of a square. Which, if any, received the greatest area of land and which received least?

Solution to July teaser

The answer provided by Suhas Shenai, UAE, is typical of that sent in by many readers. We, therefore, run his reply.

B and C cannot simultaneously have either red or yellow stamps on their foreheads, for then A's reply wouldn't be negative. C can neither have a red nor a yellow stamp on his forehead, for then B's reply wouldn't be negative, because he would have reasoned from A's reply that he does not have the same coloured stamp on his forehead as that of C. C has, therefore a green coloured stamp on his forehead. The given information cannot be used to deduce the colours of the stamp on A's or B's forehead.

Among others who gave the right answer are Kishori Lad, Bombay, Ravi Kant, Varanasi, Sandeep Harchowdhury, Dehra Dun (UP), Ramnarayan. R, Coimbatore (Tamil Nadu), Rajeev Singhal, Kullu (Himachal Pradesh), and T.B.Joseph, Bombay.

PLAYTHEMES

A KIT OF ANSWERS

T. PADMANABHAN

THE September '88 bunch of questions on words in the English language drew a tremendous response. So I tried again in April this year, and what happens? I draw virtually a blank! Anyway, here are the answers to the April questions followed by a discussion of the June questions.

1. The expansions are as follows: (a) Punjab, Afghan (border states), Kashmir, Sind and BaluchisTAN, (b) Self Contained Under Water Breathing Apparatus, (c) Women Accepted for Volunteer Emergency Service.

2. (a) a smack of ... jelly fish
(b) an unkindness of ... ravens
(c) a building of ... rooks
(d) a plague of ... locusts
(e) a mummuration of ... starlings
(f) a mustering of ... storks
(g) a murder of ... crows
(h) a school of ... fish

3. The word 'desecration' answers both parts (a) and (c). Part (b) has the obvious answer 'funeral'.

4. (a) Uncomplimentary, Unnoticeably, Subcontinental
(b) Strengths (c) Rhythms
(d) Defencelessness

5. (a) Boldface, Feedback
(b) Overnervousness
(c) Misrepresentationism, Representationism (I agree the last one is a bit shady but it is nothing compared to some of the weirdos I got on the scrabble problem!)

On now to the June issue's questions.

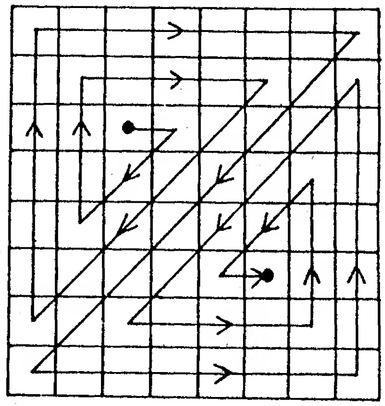
1. The first part of this question has the answer three: 7 no-trump, double, redouble. The longest sequence of bidding consists of of 316 bids! It proceeds as follows: after three passes there's a bid of one club, two passes, double, two passes, redouble, two passes. This sequence of nine bids is repeated 34 more times exhausting the 35 legal bids in bridge. After 7 no-trump redoubled, the auction terminates giving a total of 316 bids.

2. "Since you see a red side the problem is reduced to the choice between two cards. Hence the probability is ½." This is what several readers have written. But it is wrong! Even after ruling out the white-white card, there exist three different red faces in the box, any one of which could be staring at you from the table-top. (Either one of the two sides of the red-red card or the red side of the red-white card.) Out of these three, two are favourable. So the chances are 2/3. (Or, is there still a fallacy?)

3. These are the first letters of ten, twenty,... etc. It is really curious that by a mere shift you can reproduce the first letters of one, two... etc.

4. The minimum number of matchsticks needed is 27. The idea is to form a 6, 8, 10 triangle and put a horizontal row of three matches in the middle getting the shorter triangle with sides 3, 4, and 5. Since these are Pythagorean triples with the shortest perimeter, we get the minimum construction.

5. It can be accomplished with just 14 turns. See figure.



6. The southern-most tip of Europe is Punta Marroqui, a few miles south of Gibraltar; that of Africa is Cape Agulhas about 30 miles south of the Cape of Good Hope; that of South America is the Diego Ramirez islands, some 45 miles south of Cape Horn. The southern-most tip of India is in the Andaman-Nicobar islands (called Pygmalion point, I believe), and not Kanyakumari.

Best attempts were from T.N. Ram Prasad, Madras, N. Ramanathan, Kuwait, and Suhas Shenai, UAE. You will notice that I haven't answered the chess problem of the April issue. No reader attempted it; so it is still kept open. Come on, now!

FUN WITH MATHS

A GENIUS REMEMBERED

ARUN M. VAIDYA

IN the August '89 issue of **2001**, we marvelled at the mathematical genius of Pravinbhai Mehta, an unlettered man. This time we discuss another mathemagic wizard, D.R. Kaprekar, who died in 1986. Kaprekar was a frequent contributor to SCIENCE TODAY, 'Kaprekar numbers' being just one of his many discoveries in the fascinating world of numbers. He was born in 1905. Earning his graduate degree, he became a school teacher in Deolali, Maharashtra, and was attracted by the charm of numbers since his young days. He defined and studied, among others, palindromic numbers, self numbers, Dattatreya numbers, Demlo numbers, and Harshad numbers.

We begin with self numbers. Take a number, say 12. Add to this the sum of its digits. We get 15. We say 15 is gene-

rated by 12. Similarly 16 is generated by 8 and 17 by 13. But 20 is not generated by any number. Therefore 20 is a self number. The sequence of the first few self numbers is 1, 3, 5, 7, 9, 20, 31, 42, 53.

From every self number we can start a sequence of numbers in which each number is generated by the preceding one. Some of these sequences are

1, 2, 4, 8, 16, 23, 28, 38, ...
20, 22, 26, 34, 41, ...
86, 100, 101, 103, 107, ...

Poetic readers can consider each of these sequences as a river. Sometimes two rivers have a *sangam*! For instance, take the first and the third rivers above. The first river is
1, 2, 4, 8, 16, 23, 28, 38, 49, 62, 70, 77, 91, 101, 103, 107...

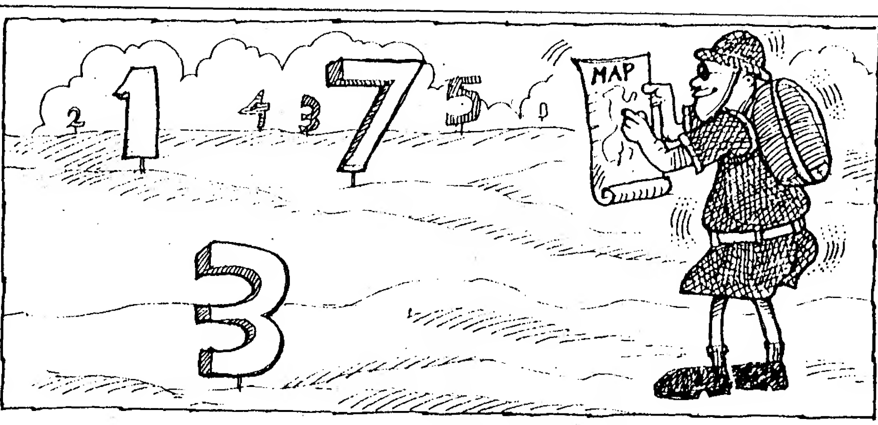
We see that it meets the third river at 101. Thus, 101 is a junction number of order 2. 103 is also a junction because it is generated by both 101 and by 92. The river starting at 1 meets one river at 101 and another at 103. Then it meets a river at 212, 214, etc. Kaprekar believed that there are only three main rivers, starting from 1, 3 and 9. Every other river will eventually meet one of these. As far as I know no one has still been able to prove or disprove this.

Are there *triveni sangams*, that is, three rivers meeting at the same point? Sure. Kaprekar gives several examples of junction numbers of order 3. But before we give an example, we will introduce an abbreviation. If the same digit is repeated several times in a number we shall shorten the number as follows:

$$155532 = 1(5)_3 32$$
$$9999337 = (9)_4(3)_2 7.$$

Now the number $1(0)_{10}001$ is generated by the three numbers:
$1(0)_{10}000$, $(9)_{10}901$ and $(9)_{10}892$.
Similarly $1(0)_{97}710$ is generated by $(9)_{97}819$, $1(0)_{97}692$ and $1(0)_{97}701$.

If you are impressed by these 101-digit numbers, Kaprekar has greater surprises in store for you. He gives a 1116-digit number which is generated by four different numbers. The number is $1(0)_{1111}1111$ and the four generators are:
$(9)_{1111}1105$, $(9)_{1111}1096$, $1(0)_{1111}1104$ and $1(0)_{1111}1095$.



BAJU PARTHAN

During the Gandhi centenary year, Kaprekar observed that Gandhiji's 100th birthday was on 2-10-1969 and 2+10+19+69=100. In 1978 he made the observation that 19+78=97, the middle two-digit number in 1978. When will this happen again, he asked. Answer: 2307.

Pravinbhai Mehta, D.R. Kaprekar and all other devotees of numbers have one thing in common: they are familiar with all the idiosyncratic properties of all the numbers. However, most of us are familiar only with the utilitarian value of numbers. It is given only to the gifted few among us to explore the mysteries of the number world.

BRAIN TEASER

A cardboard box manufacturer was considering doubling the volume of his square boxes. On hearing of this, his supplier of cardboard decided to encourage him to place the extra business by offering a very generous 37.5 per cent quantity discount on his new total turnover figure. How much extra would the box manufacturer have to pay for the additional cardboard if he decided to go ahead?

Solution to August teaser

Method: The alphabet on the top row was substituted by the random order of letters on the bottom row, thus:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S	L	U	F	X	C	O	T	I	M	Y	G	B	W	N	P	D	H	V	A	J	Z	Q	E	K	R

Hints on decoding: As the problem involves a simple substitution code, the initial method of attack should be a letter count (most people are aware that the vowels are the most frequently used letters in the English language). A count on the coded passage shows that the most frequent letters are A(18), I(13), N(19), S(13), and X(19). It would therefore be a good working supposition that these five would account for most of the vowels between them, with a reasonable certainty that either A, N or X will decode out as E.

Using this initial information as a basis, begin to experiment with possible substitutions, building up words and phrases letter by letter.

Answer:
The King, observing with judicious eyes,
The state of both his universities,
To Oxford sent a troop of horse, and why?,
That learned body wanted loyalty;
To Cambridge books, as very well discerning
How much that loyal body wanted learning.
On George I's Donation of the Bishop of Ely's library to Cambridge University by Joseph Trapp, 1679-1747).

This column has the satisfaction of posing a tough teaser at last. Only *two* readers sent in the correct answer: T.N. Ram Prasad, Madras, and Rajeev Singhal, Kullu (Himachal Pradesh). Congratulations!

PLAYTHEMES

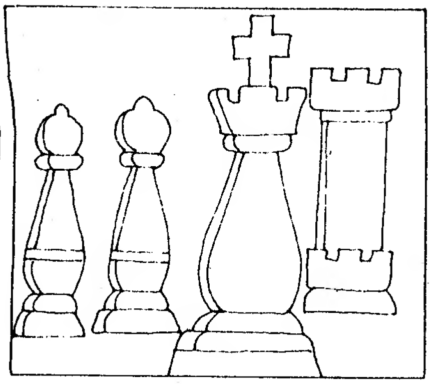
PROBABLY SPEAKING

T.PADMANABHAN

What is the probability of your getting all five answers correct?

AFTER quite a few answering sessions, here's starting on a fresh set of posers, dealing with the theory of probability. Most of them are classics, but still retain their charm because every once in a while someone comes up with a novel method of attack.

1. Our absent-minded professor buys two boxes of matches (each containing N matchsticks) and places them, one in each pocket. Every time he needs a matchstick, he randomly selects one from either box. After a while he takes out one of the boxes and finds that it is empty. (He must have absent-mindedly put the empty box back in his pocket after he had used the last matchstick.) What is the probability that at this moment there are K matches in the other box?



2. "Come on. Let's play a few games of chess."

"Well I don't know whether I want to. You play with lot of side bets."

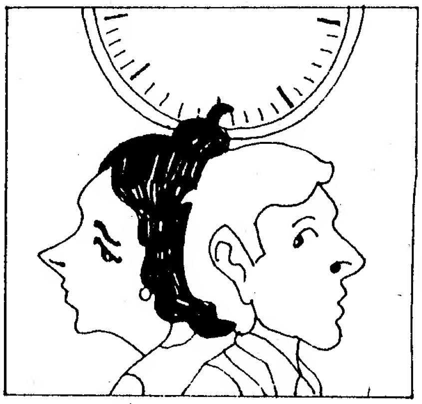
"That's part of the fun. Would you like to bet that you will win five games out of eight? After all, you know that we are equally strong players."

"Hum...m. Not really. But I'll probably bet that I will win three games out of four ?"

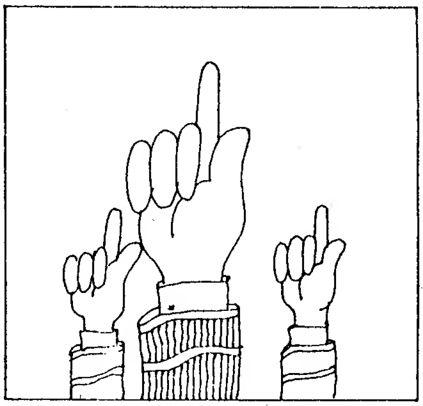
You comments, please.

3. There is a folklore that in rural Russia, girls decide about their matrimonial prospects in the following way. A girl would hold six long blades of grass in her hand with the ends protruding at the top and bottom. Another girl would tie together the six upper ends in pairs and the 6 lower ends in pairs. If, by following this procedure the blades of grass have turned into a ring then it is believed that the girl would get married within a year. What do you say are the chances for a rural Russian girl to get married?

Illustrations: Nana Shivalkar



4. He and she are supposed to meet at a shopping complex between 12 noon and 1 p.m. Whoever arrives first will wait for the other only for 15 minutes. What is the probability of the meeting taking place if each selects the moment of arrival within that interval randomly?



• 5. You must have noticed that certain decisions at the national level can be taken by a simple majority (for instance, the election of an MP). For some other decisions one requires a higher majority, say, two-thirds. One feels intuitively that a two third majority reflects the mood of the electorate more decisively than a simple majority. Let us suppose we want to be confident, at a 'confidence level' of one part in hundred, that the final decision reflects the mood of the electorate. Further, assume that there are N voters. What majority will you consider to be decisive? (Warning: This problem is more advanced than the usual ones and requires a little bit of extra knowledge.)

PLAYTHEMES

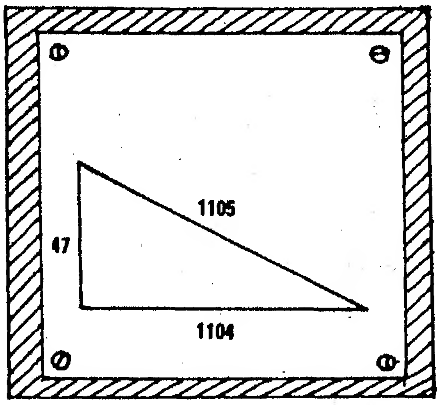
CLASSICS RECREATED

T. PADMANABHAN

One of the greatest American puzzlists, a few of Sam Loyd's puzzles had become a national craze. Here are the classic answers to his problems

Illustrations : Nana Shivalkar

THE August instalment, with the classics created by Sam Loyd, seems to have prodded several readers in the rib. Here are the answers.



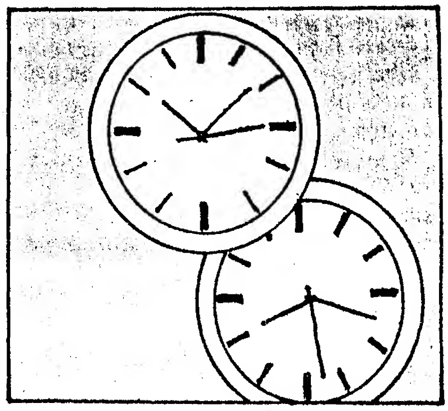
1. The right-angled triangle with one side having 47 units will have the other two sides of length 1104 and 1105. In spite of rather largish dimensions, the solution was found by several readers: Abel Lawrence (Cochin), Mohammed Nisar, Nageshwaran (Trivandrum), Prakash Narayan (Bombay), R.S. Kumar (Salem), R.M. Singhal (Kulu, Himachal Pradesh), C.G. Subramaniam (Madras), A.J. Mitra (Calcutta), S.D. Borgaonkar (Aurangabad), K. Anjan (Prakasham, A.P.), Santosh Iyer (Jaipur), Prantik Mazumder (Calcutta).

2. The first thing to realize is that we can treat the length of the army and its speed to be the basic units. So suppose that the length of the army as well as the time it takes to march its length be 1 unit; the distance travelled by the courier and his speed be x units. Then it is fairly easy to see that x satisfies the equation $x^2-2x-1=0$ which has the positive root $(1+\sqrt{2})$. Since the actual speed is 50 miles, the distance travelled by the courier is 50 $(1+\sqrt{2})$ or about 121 miles.

In the case of a square formation, the courier is travelling with a speed $(x-1)$ and $(x+1)$ relative to the moving army in his forward and backward trips and with the speed $(\sqrt{x^2-1})$ in the diagonal trips. Using this we will get a fourth degree equation: $x^4-4x^3-2x^2+4x+5=0$ which has the acceptable root 4.18. Multiplied by 50, we get the final answer to be about 209 miles. This problem is solved correctly by Mohammed Nisar, G. Nageshwaran (Trivandrum), S.K. Shenai (UAE), R.S. Kumar (Salem), Santosh Iyer (Jaipur), S.D. Borgaonkar (Aurangabad) and C.G. Subramaniam (Madras).

3. When the boats meet at, say, X, let us suppose they were 720 yards from one shore. The combined distance both boats have travelled is now equal to the width of the river. When they reach opposite shores, they have covered twice the width of the river and when they meet again at Z, say, they have covered 3 times the width of the river. Thus, each boat has now gone three times as far as they had gone when they first met. In the initial meeting one boat had gone 720 yards. So when it reaches Z it must have gone $3\times720=2160$ yards. This must be 400 yards more than the width of the river. Thus performing a mere subtraction of 400 from 2160 we get the river's width as 1760 yards. Correct answers were received from Prakash Narayan (Bombay), R.D. Mathur (Calcutta), R.M. Singhal (Kulu), P. Mazumder (Calcutta), C.G. Subramaniam (Madras), S.D. Borgaonkar (Aurangabad), Santosh Iyer (Jaipur), S.K. Shenai (UAE) and G. Nageshwaran (Trivandrum).

4. Let us suppose x is the number of dogs and also the number of dogs left among the seven animals is y then it is easy to arrive at the equation $3x = 11y + 77$. We also know that y is not more than 7. Trial and error shows that only the values 2 or 5 will make x an integer. But we also know that the rats were purchased in pairs; if y is 2 the original purchase will be 33 rats, an odd number. Hence y is 5. The merchant bought 44 dogs and 22 pairs of rats. Finally he has animals with a combined worth of 13.2 bucks. Correct answers were received from: R.M. Singhal (Kulu), Mohammed Nisar, A.J. Mitra (Calcutta), Santosh Iyer (Jaipur), S.K. Shenai (UAE), G. Nageshwaran (Trivandrum), P. Mazumder (Calcutta) and R.D. Mathur (Calcutta).



5. This was the easiest problem. The time shown by the clock will be about 10 hours, 9 minutes, and 13.85 seconds in the first case and 8 hours, 18 minutes, 27.69 seconds in the second case. G. Nageshwaran (Trivandrum), R.D. Mathur (Calcutta), S.K. Shenai (UAE), Santosh Iyer (Jaipur), Mohammed Nisar, C.G. Subramaniam (Madras), K. Anjan (Prakasham, A.P.), Able Lawrence (Cochin), and S.D. Borgaonkar (Aurangabad) have got the answers right.

6. Quite clearly, the eagle would have completed its trip in 39 sunrise to sunset periods as it sees them. But the Earth will have rotated 39 1/2 times and so 39 1/2 days would have elapsed in Washington during the flight. Nobody solved this right though C.G. Subramaniam (Madras) came pretty close.

The best answers were from C.G. Subramaniam, Santosh Iyer (Jaipur) and G. Nageshwaran (Trivandrum). Congrats!

MAGIC TRIANGLES

SHAILENDRA SHEKHAR

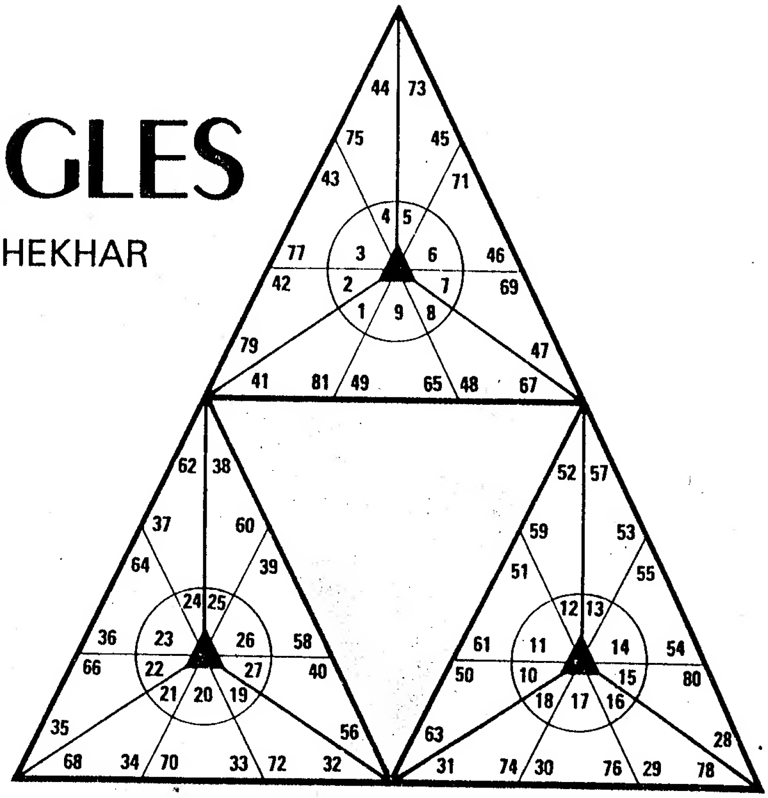


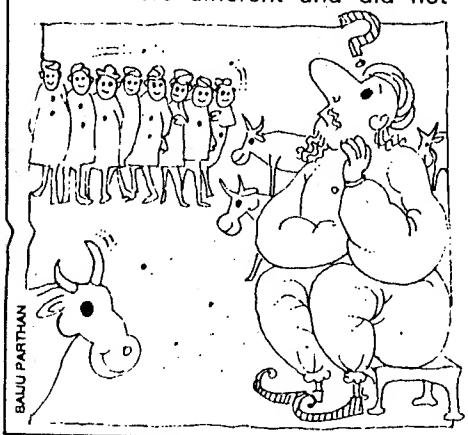
Fig. 1

FIRST, read this story carefully:

Once upon a time there was a farmer who owned 81 cows. He had numbered them according to the amount of milk they provided. For example, the cow which gave 6 litres of milk per day was called number 6 ; the cow yielding 15 litres of milk was called number 15 and so on.

When the farmer grew old, he decided to distribute the cows among his 9 sons in such a way that every son got equal number of cows, and equal amount of milk too.

His youngest son, a middle-school student, quickly figured out that all his brothers should receive 9 cows each, which provide 369 litres of milk. This much calculation was simple, but the real problem was how to divide the cows? And the problem became more complicated, when all the brothers started fighting for three particular cows of their personal choice. For example: The eldest brother insisted on cow nos 2,13, and 78; the second brother wanted cow nos 16,20, and 25; and so on. But fortunately their choices were different and did not overlap. Everybody selected 3 cows, and they were ready to get any other 6 cows.



If you were in place of the farmer what would you do? If you think you are capable of solving the problem, you need not read any further. If not, then you may like to know that there are many interesting ways to arrive at millions of answers. Yes, millions of different combinations of 9 cows are possible, and every brother can get at least 3 cows of his specific choice.

Look at Fig. 1, which shows a big equilateral triangle, divided into 4 equilateral triangles. Out of these, 3 corner ones are subdivided into three smaller thick-lined triangles; and every thick-lined triangle is further divided into three smallest thin-lined triangles; thus total number of thin-lined triangles is $3\times3\times3=27$; at each angle of which, a figure is written representing the number of a cow and the amount of milk it gives.

The numbers filled in Fig.1 are from 1 to 81 without repetition. Sum of 3 figures in every thin-lined smallest triangle is 123, in small black-lined triangle is 369. Out of these 27 thin-lined triangles; any combination of 3 can be allotted to any son. By formula of factoral number $(1 \times 2 \times 3 \times 4 \ldots \times 27)$ we can have millions of sets of 9 figures, whose total is 369, thus it is very easy to provide any three cows of one's *particular choice* to each son of the old farmer.

Another way to solve would be as in Fig. 2, which has totally different sets of numbers. Each of the nine-black-lined equilateral triangle has a small grey triangle at its nucleus, which contains 3 figures (sum=123). There are three pairs of figures (sum=82) each around the nucleus. Total sum of nine figures in a black-lined triangle is: $(82\times3=246) + 123 = 369$. All these *twenty seven pairs are exchangeable*, thus any three pairs can be fitted with any grey triangle. Besides these, a number of other combinations are possible, some of which might overlap with given solutions. The problem can be used to prepare rotatory magic squares (see also 'Magic Squares Within Magic

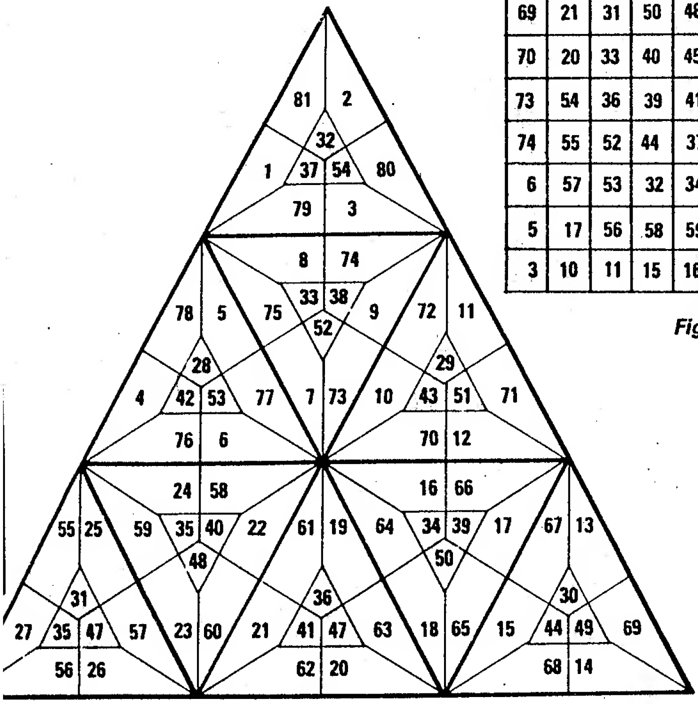


Fig.2

1	72	71	67	66	2	4	7	79
68	63	26	24	23	22	64	65	14
69	21	31	50	48	47	29	61	13
70	20	33	40	45	38	49	62	12
73	54	36	39	41	43	46	28	9
74	55	52	44	37	42	30	27	8
6	57	53	32	34	35	51	25	76
5	17	56	58	59	60	18	19	77
3	10	11	15	16	80	78	75	81

Fig.3

Squares' June 1989) in the following way. Make a central square of 3 × 3 cm, then another square of 5×5 cm on a different piece of paper. Similarly, make 7×7 cm and 9×9 cm squares too. Put smaller square on bigger square, and attach them with a pin at the centre. Fill the figures from 1 to 81, as shown in Fig.3. Sum of 9 figures in each vertical, oblique, and horizontal row is 369. Now, you can rotate any square in any desired position, but the sum will remain unchanged. Isn't it surprising?

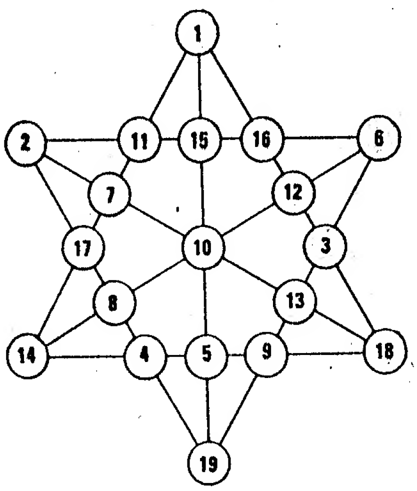
Each square can be rotated to four different positions, therefore we can get 4×4×4 = 64 different positions, each giving two answers (horizontal and vertical), thus 128 solutions in total.

On the same principle, can you device rotatory magic triangles within magic triangles?

We receive a lot of articles claiming to have solved the problem of trisecting an angle. However, it has been proved that this problem is unsolvable. So we would like to inform our readers not to send us articles on this subject. — *Ed.*

BRAIN TEASER

The accompanying magic star has a magic sum 50. Change this value to another magic number, 1990, using the minimum alterations.



K. A. GOVANDE

Solution to October teaser

1) Let the side of the box be 1 cm Therefore volume of the box = 1 cm × 1 cm × 1 cm = 1 cm³

2) To double the volume, the side of the box should be $3\sqrt{2}$=1.26 cm. Say (1.26 cm x 1.26 cm x 1.26 cm = 2 cm³)

3) A box has 6 square faces of 1 cm² each. Therefore area of cardboard required to form a box = 6×1×1=6cm²

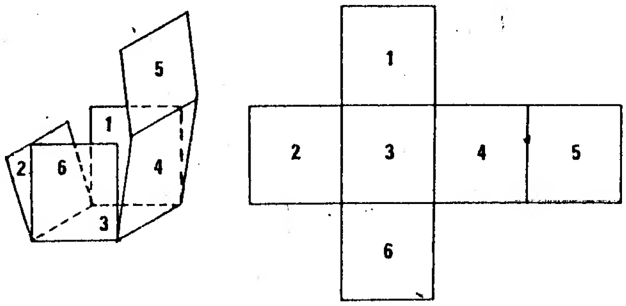
4) Area of cardboard required for new box = 6 x 1.26 x 1.26 = 9.5256 cm²

5) Therefore the cost of cardboard will be $\frac{9.5256}{6}$ =1.5876 times the original cost. That is if the earlier box cost Rs. 100, the new box will cost Rs. 158.76.

6) However, the supplier has offered 37.5 per cent discount. The manufacturer has to pay only Rs.62.5 instead of Rs. 100. Therefore, for the new box he has to pay $\frac{158.76}{1} \times \frac{62.5}{100}$ = Rs. 99.225 only.

7) He is fortunate as he has to pay Rs.0.775 less per Rs.100 even after doubling the volume of the box.

Some of the correct answers with detailed procedure were from K. Surendra Mohan, Madras, Ravindra N., Visakhapatnam, Rajeshwari Singh, New Delhi, Rajat Monga, Varanasi, R. Thireuvengadam, Madurai, Meghraj Bhatt, Valsad, and Ravikaant, B.H.U.



PLAYTHEMES

COINING PROBLEMS

PADMANABHAN

AFTER a set of answers, we go on to a set of questions. This time they deal mostly with geometry.

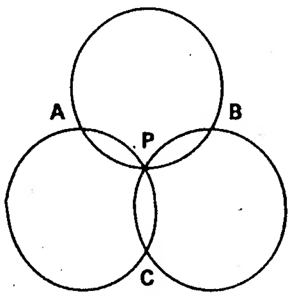


Fig. 1

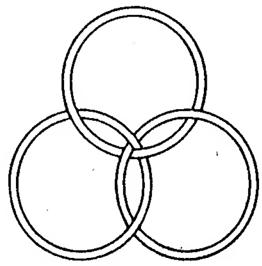


Fig. 3

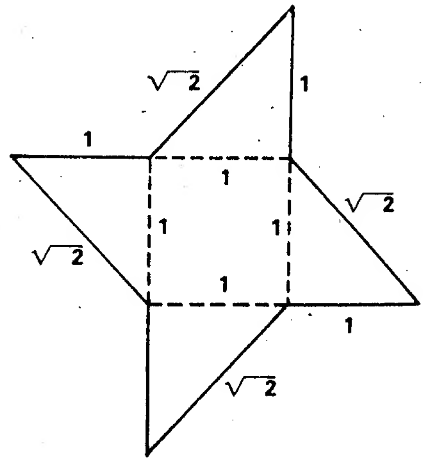


Fig. 4

1. Let us begin with a simple one. Fig. 1 shows three circles of radius 1 cm, intersecting at A, B, C and P. We now draw a circle passing through A, B and C. What will be the radius of this circle?

2. Three coins of different sizes are lying flat on the table touching one another. Strangely enough, their centres have formed a right-angled triangle. If the radii of the coins are integers, find the smallest triplet of radii which could satisfy the above condition.

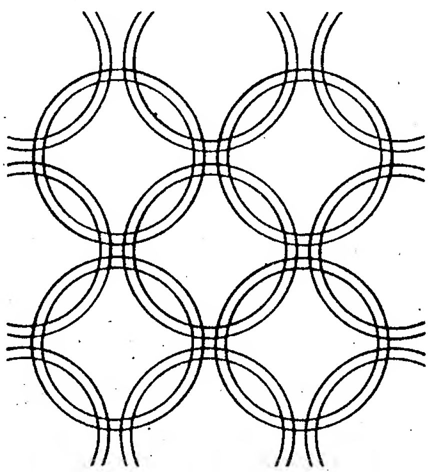


Fig. 2

3. The chain armour used by the medieval knights (called hauberk, I believe) is made by connecting the rings as shown in Fig. 2. Every ring is connected with four corners around it. The rings are actually linked together.

There is a way of connecting rings, without really linking them (see Fig. 3). Here, the three rings are connected but no two of them are linked together. If you cut any *one* ring, all the three can be separated.

Question: Can you make a hauberk using the scheme given in Fig. 3?

4. If you divide a line AB at C such that AC:CB=AB:AC, it is said to be divided in the 'golden ratio'. Numerically, this ratio happens to be $\frac{1}{2}(\sqrt{5}+1)$. A rectangle whose sides are in this ratio is supposed to be very pleasing to the eye.

Suppose you have a square paper and you want to measure off a golden rectangle. Since you have no tools at hand, you have to do this purely by folding the paper appropriately. How can you perform this origami?

5. A mathematician's table is in the crazy shape shown in Fig. 4. You are asked to place n points on this table, such that each point is separated from all others by a distance of at least $\sqrt{2}$ units. What is the maximum value of n that is possible?

6. Here is an old chestnut. Prisoners are lodged in a jail in a linear array of cells numbered 1, 2, 3 ... N, with one prisoner per cell. An eccentric jailor decides to pardon some of the prisoners in the following manner. He walks along the cells, turning the key in each door (thereby opening all the cells). He then takes a second walk turning the keys in every alternate door starting from cell 2. Then he takes a third walk, turning the keys in every *third door* starting from *cell 3, and so on.*

Analyse this procedure and decide which doors will be finally open.

FUN WITH MATHS

PRIME TIME AGAIN

S.D.BORGAONKAR

AS is familiar to all schoolkids a prime is a natural number that is divisible only by one and the number itself. The only even prime number is 2, all other primes in the number line being odd. Prime numbers are spaced irregularly, and there are an infinite number of them. Obviously every prime number established as the biggest has one bigger than itself. All other numbers can be obtained from prime numbers by addition, subtraction or multiplication. Any prime except 2 can be expressed in the form $(4n-1)$ or $(4n+1)$. While all primes can be expressed as the difference of squares of two integers, the $(4n+)$-type primes are peculiar in that they can be expressed as the *sum* of two squares. For instance, $277=14^2+9^2$; $4649=68^2+5^2$; $29=5^2+2^2$.

All primes except 2 and 5 give recurring digits on decimal expansion of their reciprocals. The number 3 has only one recurring digit and all other primes have more recurring digits which have to be found by actual division. However, it is ascertained that the number of recurring digits on decimal expansion of the reciprocal of any prime 'P' is either (P-1) or a factor of (P-1). The number of recurring digits is said to be the period or period length of the prime. For brevity I have abbreviated to RD the term 'The number representing the recurring digits on decimal expansion of the reciprocal of a prime'. If the number of digits of the RD of a prime P is (P-1) then P is called a maximal prime and if this number is a factor of (P-1) the prime is said to be a non-maximal prime. We discuss below some of the fascinating properties of RDs. We begin with maximal primes.

There are only nine maximal primes in the first 100 natural numbers. These are 7, 17, 19, 23, 29, 47, 59, 61, and 97. So we concentrate on these.

1. The sum of the digits of RD is divisible by 9. For example, RD of 7 is 142857; these digits sum up to 27. The RD of 29 is 0344827586206896551724137931. The sum is 126. When reduced further the final single-digit sum is 9.

2. When the number representing the recurring digits of P is multiplied by any number from 2 to (P-1), the product will be a cyclic variation of RD. The RD of 7 is 142857; $142857\times2=285714$; $142857\times5=714285$.

3. When the RD of P is multiplied by P, the product will be simply the number 9 repeated (P-1) times.
RD of 17=0588235294117647;
RD$\times$17=9999999999999999.
RD of 7=142857; RD$\times$7=999999.

If the number representing RD of P is multiplied by any number greater than P the product will have more than (P-1) digits; but, if to the last (P-1) digits of the product, the remaining part of the product is added the sum will be a cyclic variation of RD provided the multiplier is not divisible by P. If the multiplier is a multiple of P, the sum of the parts of the product will again be 9 repeated (P-1) times. Thus
RD of 7=143857
RD$\times$1485=212142645, that is, 212+142645=142857
RD$\times$84=11999988, that is, 999988+11=999999.

4. The RD of a maximal prime can be divided into two blocks of equal number of digits, (P-1) being even, and the sum of the two blocks will be 9 repeated as many times as the block has digits. For instance, RD of 7=142857 and 142+857=999. Similarly, RD of 19=052631578947368421
Here
052631578+947368421=999999999.

It may also be noted that the sum of the two numbers in each column of the two blocks is 9.

5. The RD can be divided into blocks of equal number of digits in many ways as in the case of 29, 61, 97 etc. The final sum of such blocks in each case will be 9 repeated as many times as the number of digits in the block. The RD of 61 has 60 digits: 016393442622950819672131147540983606557377049180327868852459.

To illustrate the property let us divide the RD in five blocks and six blocks and take their sum.

Sum of 5 blocks	Sum of 6 blocks
016393442622	0163934426
950819672131	2295081967
147540983606	2131147540
557377049180	9836065573
327868852459	7704918032
1999999999998	7868852459
+1	29999999997
999999999999	+2
	9999999999.

Thus the final sum in each case is 9 repeated 12 and 10 times respectively.

6. When the RD can be divided into an even number of blocks of the same number of digits, the sum of the digits in each column of the blocks is the same, a multiple of 9. For example the RD of 29 is of 28 digits: 0344827586206896551724137931.
Sum of its four blocks is
0344827
5862068
9655172
4137931
19999998

Thus the final sum is 9999998+1=9999999.

If the RD is divided into an odd number of blocks, the final sum will be of 9 s, but the sum of the digits in each column will not be the same. (Exception: Each column adding up to 9)

It should be noted that in forming blocks, zero (or zeroes) at the top are not to be omitted as they are significant.

Non-maximal Primes

Consider now the properties of the RDs of non-maximal primes. The period length of a non-maximal prime P is a factor of (P–1) say K = (P–1)/n.

1. If RD of P is multiplied by P the product will be the number 9 repeated K times. For instance, RD of 13 is 076923; 076923x13=999999.

2. If RD of P is multiplied by any number less than P, the product will be a cyclic variation of RD or (n-1) other variants which are multiples of RD.

The RD of 13 is 076923 of 6 digits i.e. 1/2 (13-1). Hence the RD must have one variant which is 153846. We have 076923x3=230769; 076923x7=538461.

Thus the products are cyclic variations of RD and its variation respectively.

The RD of 41 is 02439, having five digits. 1/8(41-1). Hence the RD must have seven variants which are 04878, 07317, 09756, 12195, 04634, 26829 and 36585. If the RD 02439 is multiplied by any number less than 41, the product will be a cyclic variation of RD.

3. If RD is multiplied by any number greater than P, but not a multiple of P, the product will be a cyclic variation of one of the n forms — RD and its (n-1) variants. If the product has more digits than K — the period length of P — the product has to be reduced to K digits. Thus for RD of 13 being 076923,
076923x15=11534845;
153845+1=153846, the other variant of RD.
076923x16=1230768;
230768+1 = 230769, a cyclic variation of RD.

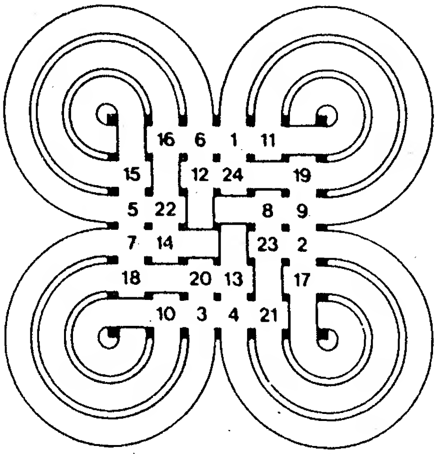
4. If the RD of P is multiplied by a multiple of P the product reduced to K digits will be 9 repeated K times. RD of 13 is 076923; 076923×104=7999992 gives 999992+7=999999.

5. The RDs of non-maximal primes have distinct variants and they have the same properties as those of their corresponding RDs. For example the RD of 73 is 01369863 and it has eight distinct variants: 02739726, 05479452, 06849315, 08219178, 09589041, 10958904, 12328767 and 15068493. Multiply one of the variants by 73; 10958904×73=79999992, that is 99999999.

6. The properties of blocks of equal digits of RDs of non-maximal primes and their variants are the same as those of maximal primes.

7. We have seen that a non-maximal prime having period length K where K equals (P-1)/n, has (n-1) variants of RD. It is amusing to note that if RD or any variant is multiplied by 1 to (P-1) there will be exactly K numbers giving products conforming to RD and its other (n-1) variants.

BRAIN TEASER



The numbers 1 to 24 inclusive have been positioned on the cross-over areas of a woven triple-strand design.

Twelve of the cross-overs of the woven design have been completed. Mark in the remaining 24 cross-overs in such a way that each of the three strands ends up with eight numbers totalling exactly 100.

Solution to November teaser

The six numbers are 784 847 478 487 874 and 748. Omitting 784, the other numbers add up to 3434. Let ABC be the three digits (all different) and the six numbers be written as ABC, ACB, BCA, CBA and CAB and BAC. (The value of ABC in decimal notation is 100A+10B+C).

If all six numbers are added their sum S is given by S222×(A+B+C)222×K where K = (A+B+C). If one number (say ABC) is omitted and the other five are added to give a total T, then T = 122A+212B+221C and S-T will give the value of the number omitted.

Now 2T = 244A+424B+442C
= 243A+423B+441C+(A+B+C)
or 2T = 9×(27A+47B+49C)+K

From this we can deduce that 2T and K give the same remainder when divided by 9. Hence, their digital roots are the same. If R is the digital root of 2T, then K=R or R+9 or R+18 and correspondingly S = R×222 or (R+9)×222 or (R+18)×222.

Since S>T and S-T<1000 we choose that value of K to obtain the appropriate value of S. Then S-T will give the number omitted and by permuting the digits the other five numbers can be obtained.

As a check the digits in the answer obtained are added and matched with the corresponding K. If they match, the solution is correct. Using this we can test each value of T from 3431 to 3439. Say T=3431; then the digital root of T is 2, and the digital root of 2T is 4. K = 4, 13 or 22; S888, 2886 or 4884. In the first two cases S<T, in the third S-T<1000. All these values of S are inadmissible. Therefore, T=3431. Now suppose T=3433. Then digital root of T is 4 and digital root of 2T is 8. Hence, K=8, 17 or 26. And S=1776, 3774 or 5772. Here S<T, S-T=341 and S-T>100. But in the second case 3+4+1=8 and K=17. So T=3433. We thus find that only T = 3434 yields 784 as the number omitted.

This teaser too was apparently a sitter for many of the readers, judging from the number of answers we received. The first correct were from Ravindra N., Visakhapatnam, Suneet Kumar, New Delhi, Prof. K.D. Kadam, Pimpalner (Maharashtra), Shashi Bhushan, Dhanbad, Prabhat Kumar, Patiala, K. Kumar, Trivandrum, Manoj Kumar Agarwal, Bhubaneshwar, Nalini Gujar, New Delhi, Mahesh Patil, Thane (Maharashtra), and K. Anandakumar, Coimbatore.

Many of the readers solved the problem by the trial and error method involving lengthy computations. Among the few logically presented answers was the one from J.H.Bhatt, Baroda. Some tried out the problem on computers and came up with the correct answer. A few of these have even sent us the program which they used to obtain the solution. Think of new ways to solve old problems!